INTRODUCTION

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Differential Equations

Classification of Differential Equations

- ODE and PDE
- Order and Degree of Differential Equations
- Linear Differential Equations

Applications of ODEs

- Population models
- Predator-prey models
- Electrical circuits
- Bessel's equation

Solution of Differential Equations

The Elimination of Arbitrary Constant

Families of Curves

This course is concerned with differential equations (DEs) and their solutions methods. Methods will be employed to obtain approximate solutions of the differential equations which consider the equation $y = y(x)$, which describes a relationship between two quantities. The **dependent** variable y, and the **independent** variable x.

Definition

A *differential equation* is an equation containing an unknown function and its derivatives.

Example (1)

- $\frac{dy}{dx} = 5x + 2$
- $4\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = ay$
- $\left(\frac{d^{100}y}{dx^{100}}\right)^{99} + \left(\frac{dy}{dx}\right)^{101} = 3,$

we show that in this example, y is *depentent* variable and x is *independent* variable. These equations are called *ordinary differentail equations (ODE)*.

Example (2)

- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$
- $3\frac{\partial^4 u}{\partial x^4} + 4\frac{\partial^2 u}{\partial t^2} = 0.$

In the previous example, u is *dependent* variable and x and y are *independent* variables. These equations are called *partial differential equations(PDE)*.

ODE and PDE

Definition

We say that an equation is ODE if the unknown function depends only on one variable.

Example

- $\frac{dy}{dx} = 5x + 2.$
- $4\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = ay.$
- $\left(\frac{d^{100}y}{dx^{100}}\right)^{99} + \left(\frac{dy}{dx}\right)^{101} = 3.$

Definition

We say that an equation is PDE if the unknown function depends on more than one variable.

Example

- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$
- $3\frac{\partial^4 u}{\partial x^4} + 4\frac{\partial^2 u}{\partial t^2} = 0.$

An equation, or systems of equations, may involve more than one dependent variables, but provided only a single independent variable appears then it is still termed ordinary. For example,

$$u' = 2u + 3v^2; v'' = u^2 + 4v;$$

where

$$u = u(x) \text{ and } v = v(x),$$

is a system of ODEs since there is only one independent variable x.

Table: Note

$\frac{dy}{dx} = y^{'}$	$\frac{d^2y}{dx^2} = y^{''}$	$\ldots$	$\frac{d^ny}{dx^n} = y^n$
$\left(\frac{dy}{dx}\right)^2 = (y^{'})^2$	$\left(\frac{dy}{dx}\right)^3 = (y^{'})^3$	$\ldots$	$\left(\frac{dy}{dx}\right)^n = (y^{'})^n$

Order and Degree of Differential Equations

Definition

1. The **order** of the differential equation is order of the highest derivative in the differential equation.
2. The **degree** of a differential equation is given by the degree of the power of the highest derivative used.

Table: Examples of the **order** and the **degree** of the differential equations

Differential Equation	Order	Degree
$y'' + 2y' + y = 0$	2	1
$(y')^2 + 3y = x + 2$	1	2
$y''' + y = xe^8$	3	1
$(y'''')^8 + (y')^{20} = 0$	4	8

Linear Differential Equations

Definition

An ordinary differential equation of order n is called linear if it might be on the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \tag{1}$$

or

$$f(x, y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}, \ldots, \frac{d^{n-1} y}{dx^{n-1}}, \frac{d^n y}{dx^n}) = g(x),$$

or

$$f(x, y, y', y'', \ldots, y^{n-1}, y^n) = g(x),$$

where the coefficient functions $a_n, a_{n-1} \ldots a_0$ in (1), and $g(x)$ depend only on x, and all of the terms involving the dependent variable (y or its derivatives) appear in simple algebraic form.

If the coefficient functions $a_n(x), a_{n-1}(x)$, etc, are constants, then the equation is called linear with constant coefficients. If the equation can't be written as (1), then **the ODE is non-linear**.

Exercises

Classify the following equations as either linear or nonlinear, state the order and identify the dependent and independent variables.

(a) $\frac{d^3y}{dx^3} - \frac{dy}{dx} = \sin x$ (b) $\frac{d^2u}{dt^2} - \frac{du}{dt} = e^u$

(c) $\left(\frac{d^3\theta}{dz^3}\right)^2 - z\frac{d\theta}{dz} = 0$ (d) $\frac{d^2y}{dt^2} - y^3 = 1 - t$

Answers

(a) ....., third order, dependent: y, independent: x.

(b) Nonlinear, second order, dependent: ..., independent: t.

(c) ....., ..., dependent: ....., independent: .....

(d)....., ..., dependent: ....., independent: .....

Applications of ODEs

There are countless applications that are modelled by ODEs. Pioneers in developing ODEs include Leibniz (1646–1716), Newton (1643–1727) and Euler (1707–1783).

Population models

For some population $P = P(t)$, we might assume that the rate of increase or decrease is proportional to the current population

$$\frac{dP}{dt} = \lambda P,$$

which is used to model populations of biological cells, species and radioactive atoms.

Another model is

$$\frac{dP}{dt} = rP - aP^2,$$

which is called the Logistic equation. In this model the population does not increase without bound forever, but is limited at large populations.

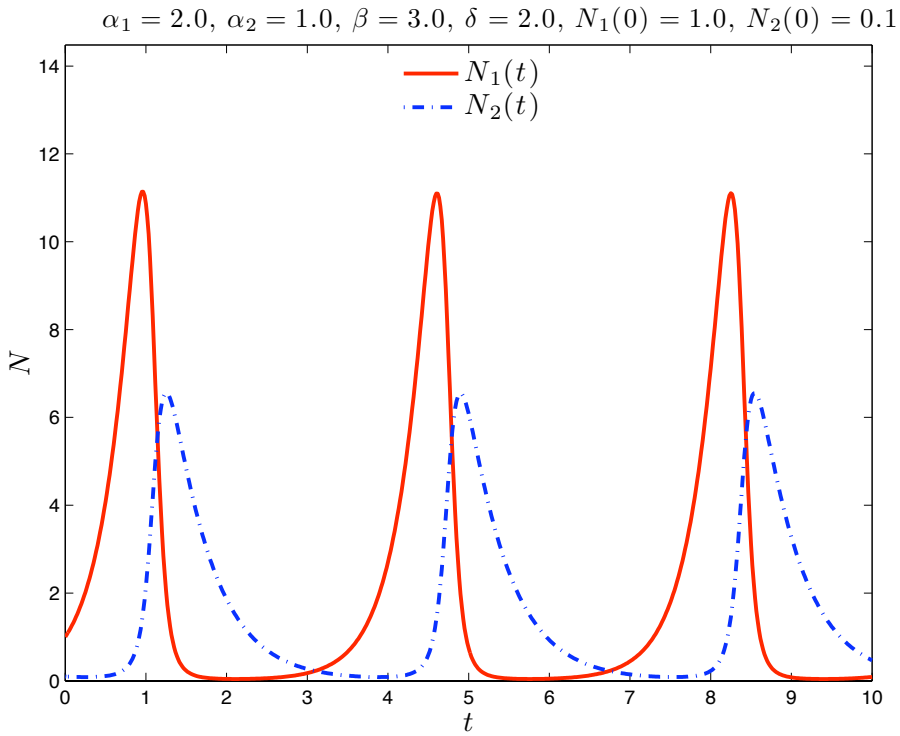
Predator-prey models

For some prey population $N_1 = N_1(t)$, and predator population $N_2 = N_2(t)$, we have the model

$$\frac{dN_1}{dt} = N_1(\alpha_1 - \beta N_2), \quad \frac{dN_2}{dt} = N_2(-\alpha_2 + \delta N_1),$$

called the Lotka-Volterra equations. This model was developed in the 1920s and was originally used to model populations of sharks and prey fish in the Adriatic Sea. Other examples of predator-prey populations include foxes and rabbits, ladybirds and aphids.

An interesting feature of the solution to this model is that the two populations can oscillate on a coupled periodic basis.


$\alpha_1 = 2.0,\ \alpha_2 = 1.0,\ \beta = 3.0,\ \delta = 2.0,\ N_1(0) = 1.0,\ N_2(0) = 0.1$
$N_1(t)$
$N_2(t)$
N
t
0
2
4
6
8
10
12
14
0
1
2
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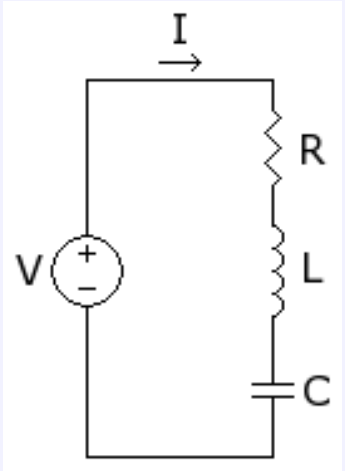
Electrical circuits



An RLC circuit comprises an resistor, inductor and capacitor in series with a cell supplying a voltage $V(t)$. The equation for the current $I(t)$ is given by

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dV}{dt}, \tag{2}$$

where R is the resistance, L is the inductance and C is the capacitance.

Equation (2) is a second order ODE with constant coefficients and forcing and it is the same equation that describes a mass-spring-damper system

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t),$$

where $x(t)$ is the position of the mass at time t, m is the mass, c is the damping constant, k is the spring constant and $F(t)$ is the applied force. Thus RLC circuits and mass-spring-damper systems exhibit similar phenomena, such as resonance and damping frequencies, which can be solved for in terms of the parameters of the system.

Bessel's equation

The second-order ODE

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-p^2)y=0,$$

is called Bessel's equation and it has many applications in heat-diffusion, electromagnetics and elasticity. The two linearly-independent solutions of this equations are

- $J_p(x)$ Bessel function of the first kind of order p
- $Y_p(x)$ Bessel function of the second kind of order p

Due to the importance of this ODE, many analytical and asymptotic properties of Bessel functions have been tabulated and recorded.

Solution of Differential Equations

we can say that the Solution of the Differential Equations (DEs) means finding an equation with no derivatives that satisfies the given DE.

Example

Prove that $y = e^{2x}$ is a solution of the equation

$$y'' + y' - 6y = 0$$

Solution

to prove that we substitute $y = e^{2x}$, $y' = 2e^{2x}$ and $y'' = 4e^{2x}$ in the equation, thus

$$y'' + y' - 6y = 4e^{2x} + 2e^{2x} - 6e^{2x} = 0 \quad \text{for all } x \in \mathbb{R}$$

Example

Verify that $y = x^3e^x$ is a solution of the differential equation

$$xy'' - 2(x+1)y' + (x+2)y = 0; x > 0$$

Solution By substituting $y = x^3e^x$, $y' = 3x^2e^x + x^3e^x$ and $y'' = 6xe^x + 6x^2e^x + x^3e^x$ in the equation thus,

$$\begin{aligned} xy'' & -2(x+1)y' + (x+2)y \\ = & \ 6x^2e^x + 6x^3e^x + x^4e^x - 6x^3e^x - 2x^4e^x - 6x^2e^x - 2x^3e^x + x^4e^x + x^3e^x \\ = & \ 0 \end{aligned}$$

Example

verify that $F(x,y) = x^2 + y^2 - 25 = 0$ satisfies an implicit solution of the differential equation $\frac{dy}{dx} = -\frac{x}{y}$, on the interval $-5 < x < 5$.

Note: we say a relation $F(x,y) = 0$ is an **implicit solution** of an ordinary differential equation on an interval I if the relation defines implicitly a function $y = \phi(x)$ satisfy the differential equation.

Solution

by using the implicit differentiation we obtain that $2x + 2yy' = 0$ or $\frac{dy}{dx} = -\frac{x}{y}$; $y \neq 0$.
We write y explicitly as a function of x. Thus

$$y^2 + x^2 = 25 \;\Rightarrow\; y = \pm\sqrt{25 - x^2},$$

then the two explicit solutions defined on the interval $(-5, 5)$

The Elimination of Arbitrary Constant

Creating a differential equation by eliminate the arbitrary constants:

Example (1)

Eliminate the arbitrary constants c from the equation

$$y = c\sin x \quad (1)$$

Solution

Since one constant is to be eliminated, we obtain the one derivative

$$y' = c\cos x$$

$$\Rightarrow \quad c = \frac{y'}{\cos x}$$

by substitution the value of c into (1) we have

$$y = \frac{y'}{\cos x}\sin x$$

$$\frac{y}{y'} = \tan x$$

Example (2)

Eliminate the arbitrary constants a from the equation

$$y^2 = 4ax.$$

Solution Since one constant is to be eliminated, we obtain the one derivative and by using the implicit differentiation we have

$$2yy' = 4a$$

$$a = \frac{yy'}{2} \quad (2)$$

by substitution from (2) into (1) we have the differential equation of the first order and the first degree

$$2xy' - y = 0$$

Example (3)

Eliminate the arbitrary constants c_1 and c_2 from the relation

$$y = c_1 e^{-2x} + c_2 e^{3x} \quad (1)$$

Solution

Since one constant is to be eliminated, we obtain the two derivative

$$y' = -2c_1 e^{-2x} + 3c_2 e^{3x} \quad (2)$$

$$y'' = 4c_1 e^{-2x} + 9c_2 e^{3x} \quad (3)$$

the elimination of c_1 from (2) and (3) yields

$$y'' + 2y' = 15c_2 e^{3x} \quad (4)$$

the elimination of c_1 from (1) and (2) yields

$$y' + 2y = 5c_2 e^{3x} \quad (5)$$

the elimination of c_2 from (4) and (5) yeilds

$$y'' - y' - 6y = 0$$

Example (4)

Eliminate the arbitrary constant a from the equation

$$(x-a)^2+y^2=a^2$$

Solution

$$2(x-a)+2yy'=0$$

$$\Rightarrow \quad a=2yy'+x$$

thus,

$$(yy')^2+y^2=(x+yy')^2$$

$$\Rightarrow \quad 2xyy'+x^2-y^2=0$$

Example (5)

Eliminate B and α from the relation

$$x = B\cos(\omega t + \alpha).$$

Solution

in this example we have B and α are arbitrary constants and ω is a fixed constant. First we obtain two derivatives of x with respect to t

$$x' = -\omega B\sin(\omega t + \alpha)$$

$$x'' = -\omega^2 B\cos(\omega t + \alpha)$$

by comparison of equation and shows at once that

$$x'' + \omega^2 x = 0$$

Exercise

Eliminate the arbitrary constant c from the family of curves

$$cxy + c^2 x + 4 = 0.$$

Families of Curves

An equation involving a parameter, also one both of the coordinates of a plane, might represent a family of curve corresponding to each value of the parameter. For example the equation

$$(x-c)^2+(y-c)^2=2c^2 \tag{3}$$

represents a family of circles with their centers on $y=x$. If we assume that c in the equation (3) is arbitrary constant, then by using the elimination of arbitrary constant, then result equation is called differential equation of the family of curve (3).
We can rearrange equation (3) to be the following

$$x^2+y^2-2c(x+y)=0$$

$$\frac{x^2+y^2}{x+y}=2c, \quad y\neq -x.$$

Families of Curves

Example (1)

Find a differential equation satisfied by the family of parabolas having their vertices at the origin and their foci (focus) on the $y-axis$.

Solution An equation of this family of parabolas is

$$y = ax^2,$$

where $a \neq 0$ is a arbitrary constant Thus

$$y' = 2ax$$

$$\Rightarrow \quad xy' - 2y = 0,$$

which is the differential equation of the family, and it is a first-order linear differential equation.

Families of Curves

Example (2)

Find a differential equation of the family of Find circles having their centers on the $y-axis$.

Solution we have to deal with the two parameters family because of a member of the family of circles of this examples have its center anywhere on the $y-axis$ and its radius of any magnitude, thus

$$x^2+(y-b)^2=c^2,$$

now we will eliminate the both b and c parameters and we will have a non-linear second-order differential equation as blew

$$xy''-(y')^3-y'=0.$$

Differential Equation of First Order

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First Order Differential Equation

- Initial Value Problem (IVP)
- Existence of a Unique Solution
- Separable Equations
- Equations With Homogeneous Coefficients
- Homogeneous Equations
  - General Method
  - Homogeneous Equations Requiring a Change of Variables
- Exact Differential Equations
- Integrating Factor
- The General Solution of a Linear Differential Equations
- Bernoulli's Equation

Here we will start to study some methods which might use to solve first order differential equations.
Consider the equation of order one

$$F(x, y, y') = 0 \tag{1}$$

We suppose that the equation (1) can be written as the form

$$y' = \frac{dy}{dx} = f(x, y). \tag{2}$$

The equation (2) can be written as follows

$$M(x, y)dx + N(x, y)dy = 0, \tag{3}$$

where M and N are two functions of x and y.

Initial Value Problem (IVP)

We are interested in problems in which we seek a solution $y(x)$ of differential equation which satisfies some conditions imposed on the unknown $y(x)$ or its derivatives. On some interval I containing x_0, the problem

$$\begin{aligned} \text{Solve:} \quad & \frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \\ \text{Subject to:} \quad & y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}, \end{aligned}$$

where $y_0, y_1, \ldots, y_{n-1}$ are arbitrary specified real constants, is clled an *initial-values problem (IVP)* and its $n-1$ derivatives at a single point x_0: $y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$ are called *initial conditions*.

Special cases

First and second-order **IVPs**

$$\begin{aligned} \text{Solve:} \quad & \frac{dy}{dx} = f(x, y) \\ \text{Subject to:} \quad & y(x_0) = y_0. \end{aligned}$$

$$\begin{aligned} \text{Solve:} \quad & \frac{d^2y}{dx^2} = f(x, y, y') \\ \text{Subject to:} \quad & y(x_0) = y_0, y'(x_0) = y_1. \end{aligned}$$

Example (1)

Solve the initial value problem $y' = 10 - x$

Subject to: $y(0) = -1$.

Solution

$$y' = 10 - x \rightarrow \frac{dy}{dx} = 10 - x \rightarrow dy = (10 - x)dx$$

by integrating both sides with respect to x

$$\Rightarrow y = 10x - \frac{x^2}{2} + c$$

Now by using the initial data, plug it into the general solution and solve for c

$$\Rightarrow -1 = 10(0) - \frac{(0)^2}{2} + c \Rightarrow c = -1$$

$$\therefore \text{solution: } y = 10x - \frac{x^2}{2} - 1$$

Example (2)

Solve the initial value problem

$$\frac{dy}{dx} = 9x^2 - 4x + 5, \quad y(-1) = 0$$

Solution

Step 1:

$\frac{dy}{dx} = 9x^2 - 4x + 5 \Rightarrow dy = (9x^2 - 4x + 5)dx$

$\int dy = \int (9x^2 - 4x + 5)dx \Rightarrow y = 3x^3 - 2x^2 + 5x + c$

Step 2: When $x = -1$, $y = 0$,

$0 = 3(-1)^3 - 2(-1)^2 + 5(-1) + c \Rightarrow 0 = -3 - 2 - 5 + c \Rightarrow c = 10$

The solution is: $y = 3x^3 - 2x^2 + 5x + 10$

Exercises

Exercise 1

Solve the initial value problem $y' + 2xy^2 = 0$

Subject to: $y(0) = -1$.

Exercise 2

Solve the initial value problem $y'' = x$

Subject to: $y(0) = 1, y'(0) = -1$.

Existence of a Unique Solution

Theorem

Consider a first order differential equation

$$\frac{dy}{dx} = f(x, y), \text{ with the initial value } y(x_0) = y_0,$$

there exists a unique solution if

- $f(x, y)$ *and* $\frac{\partial f(x,y)}{\partial y}$ *are continuous with in the region* $\mathbb{R}^2$ *of* xy*-plane.*
- (x_0, y_0) *be a point in the region* $\mathbb{R}^2$

Example

Find the largest region of the xy-plane for which the following initial value problems have unique solutions:

(a) $\sqrt{x^2-4}y' = 1+\sin(x)\ln(y)$, with initial condition $y(3)=4$.

Solution

$$y' = \frac{1+\sin(x)\ln y}{\sqrt{x^2-4}} = f(x,y)$$

$$y' = \frac{1}{\sqrt{x^2-4}} + \frac{\sin(x)}{\sqrt{x^2-4}}\ln y; \quad y>0 \quad \text{and} \quad |x|>2$$

$$\frac{\partial f}{\partial y} = \frac{\sin x}{\sqrt{x^2-4}}\frac{1}{y}.$$

Then f and $\frac{\partial f}{\partial y}$ are continuous on

$$R = \{(x,y)\in\mathbb{R}^2;\ |x|>2,\ y>0\}$$

$$R_1 = \{(x,y)\in\mathbb{R}^2;\ x>2,\ y>0\} \cup R_2 = \{(x,y)\in\mathbb{R}^2;\ x<-2,\ y>0\}$$

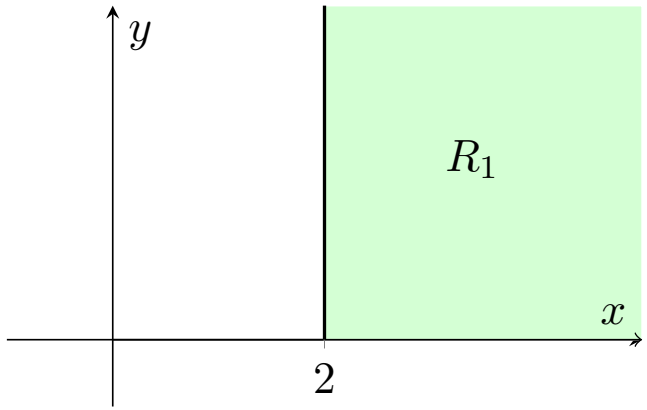


Figure: Largest Region in $xy-$plane for IVP $(3, 4)$

As we see the point $(3, 4) \in R_1 = \{(x, y);\ x > 2,\ y > 0\}$, so the largest region in xy-plane for which the **IVP** has a unique solution is R_1. If we take any rectangular R_2 with center $(3, 4)$ such that $R_2 \subset R_1$, then the **IVP** has also a unique solution, but R_2 is not the largest region.

(b) $\ln(x-2)\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{y-2}$, with initial condition $y\left(\dfrac{5}{2}\right) = 4.$

Solution we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{y-2}}{\ln(x-2)} = f(x,y)$$

by taking the derivative of $f(x,y)$ with respect to y, thus

$$\frac{\partial f}{\partial y} = \frac{1}{\ln(x-2)}\frac{1}{2\sqrt{y-2}}$$

Then f and $\frac{\partial f}{\partial y}$ are continuous on

$$R = \{(x,y) \in \mathbb{R}^2;\ x \neq 2,\ x \neq 3,\ y > 2\}.$$

But

$$R = R_1 = \{(x, y) \in \mathbb{R}^2;\ 2 < x < 3,\ y > 2\}$$
$$\cup\ R_2 = \{(x, y) \in \mathbb{R}^2;\ x > 3,\ y > 2\},$$

As

$$\left(\frac{5}{2}, 4\right) \in R_1 = \{(x, y) \in \mathbb{R}^2;\ 2 < x < 3,\ y > 2\},$$

then the largest region in xy-plane for which the **IVP** has a unique solution is R_1.

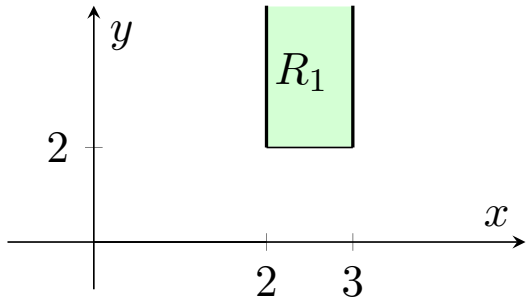


Figure: Largest Region in $xy-$plane for IVP $(\frac{5}{2}, 4)$

(c) $\sqrt{x/y}y' = \cos(x+y);\ y \neq 0,$ with initial condition $y(1) = 1.$

Solution we have

$$y' = \cos(x+y)\left(\frac{x}{y}\right)^{-1/2} = f(x,y),$$

thus,

$$\frac{\partial f}{\partial y} = -\sin(x,y)\left(\frac{x}{y}\right)^{-1/2} - (1/2)\cos(x+y)\left(\frac{x}{y}\right)^{-3/2}\left(\frac{-x}{y^2}\right)'$$

so f and $\frac{\partial f}{\partial y}$ are continuous on

$$R = \{(x,y);\ (x/y) > 0\}.$$

or

$$R = R_1 = \{(x,y) \in \mathbb{R}^2;\ x < 0 \text{ and} y < 0\}$$

$$\cup R_2 = \{(x,y) \in \mathbb{R}^2;\ x > 0 \text{ and} y > 0\}.$$

But

$$(1,1) \in R_2 = \{(x,y);\ x > 0 \text{ and} y > 0\},$$

then the largest region in xy-plane for which the **IVP** has a unique solution is R_2.

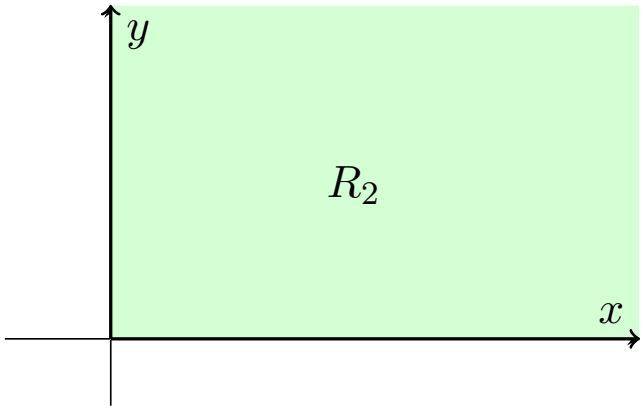


Figure: Largest Region in $xy-$plane for IVP $(1,1)$

Exercise

Determine the largest region of the xy-plane for which the following initial value problem has a unique solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y+2x}{y-2x}, \text{with initial condition } \; y(1) = 0.$$

Separable Equations

We begin to study the methods for solving the first-order differential equations. Consider a first-order differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0, \tag{4}$$

where M and N are two functions of x and y. Sometimes we can write the equation (4) as

$$F(x)dx + G(y)dy = 0, \tag{5}$$

which is said to be variables separable equation. We solve a variables separable equation by **separating** the variables and integrating.

$$\frac{dy}{G(y)} = f(x)\ dx \Rightarrow \int \frac{dy}{G(y)} = \int f(x)\ dx + c$$

Since we have one arbitrary constant in the solution, we have found the general solution of the variables separable equation.

Separable Equations

Example

Solve the following differential equations:

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x$

Solution we can separate the variables of the equation to be

$$dy = 2xdx$$

by integrating the both sides

$$\int dy = \int 2xdx$$

thus,

$$y = x^2 + c.$$

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2xy$

Solution we can separate the variables of the equation to be

$$\frac{dy}{y} = 2xdx$$

by integrating the both sides

$$\int \frac{dy}{y} = \int 2xdx$$

thus,

$$\ln|y| = x^2 + c.$$

(c) $e^x \cos y \, dx + (1 + e^x) \sin y \, dy = 0$

Solution we can separate the variables of the equation to be

$$\frac{\sin y}{\cos y} dy + \frac{e^x}{1 + e^x} dx = 0$$

$$\tan y \, dy + \frac{e^x}{1 + e^x} dx = 0$$

by integrating we have

$$\int \tan y \, dy + \int \frac{e^x}{1 + e^x} dx = c$$

thus,

$$\ln |1 + e^x| + \ln |\sec y| = \ln c_1,$$

$$\ln(|(1 + e^x) \sec y|) = \ln c_1,$$

by taking the Exponential for both side, and from the properties of exponential and logarithmic equations, thus

$$(1 + e^x) \sec y = c_1.$$

We have found the general solution of the variables separable equation. Now Find the particular solution at the point $(0, 0)$. So we have

$$(1 + e^0) \sec 0 = c_1. \qquad (\sec \theta = 1/\cos \theta)$$

$$c_1 = 2$$

thus, the particular solution is

$$(1 + e^x) \sec y = 2.$$

(d) $2x(y^2+y)dx+(x^2-1)ydy=0, \quad y\neq 0$

Solution we can separate the variables of the equation to be

$$\frac{2x}{x^2-1}dx=\frac{-1}{y+1}dy$$

by integrating the both sides

$$\int\frac{2x}{x^2-1}dx=\int\frac{-1}{y+1}dy$$

thus,

$$\ln|x^2-1|=-\ln|y+1|+c$$
$$\ln|x^2-1|+\ln|y+1|=c$$
$$\ln|(x^2-1)(y+1)|=c$$
$$|(x^2-1)(y+1)|=e^c$$
$$|(x^2-1)(y+1)|=c_1.$$

(e) $(xy + x)dx = (x^2y^2 + x^2 + y^2 + 1)dy = 0$

Solution we have

$$x(y + 1)dx = x^2(y^2 + 1) + (y^2 + 1)dy$$

$$x(y + 1)dx = (x^2 + 1)(y^2 + 1)dy$$

we can separate the variables of the equation to be

$$\frac{x}{x^2 + 1}dx = \frac{y^2 + 1}{y + 1}dy$$

by integrating the both sides

$$\int \frac{x}{x^2 + 1}dx = \int \frac{y^2 + 1}{y + 1}dy + c$$

thus,

$$(1/2)\ln\left(x^2 + 1\right) - (1/2)y^2 + y - 2\ln(y + 1) = c.$$

Separable Equations

Exercises

Solve the following differential equations:

1. $\dfrac{dy}{dx} = \dfrac{y(1-y^2)}{x(1-x^2)}$
2. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x(1-y^2)}{y(1-x^2)}$
3. $(x-1)\dfrac{\mathrm{d}y}{\mathrm{d}x} = x(y+1)$
4. $y\ln x dx + (1+2y)dy = 0$
5. $e^{x+y}\dfrac{\mathrm{d}y}{\mathrm{d}x} = e^{2x-y}$

Equations With Homogeneous Coefficients

Definition

A function $F(x, y)$ is called **homogeneous of degree** n if

$$F(tx, ty) = t^n F(x, y), \quad \text{for all } t > 0; t \in \mathbb{R}.$$

A first-order differential equation form

$$M(x, y)dx + N(x, y)dy, \tag{6}$$

is said to be homogeneous if both coefficient functions M and N are homogeneous equations of the **same** degree.
In other words, (6) is homogeneous if

$$M(tx, ty) = t^n M(x, y) \text{ and } N(tx, ty) = t^n N(x, y).$$

Example

1. If $M(x,y)$ and $N(x,y)$ are both homogeneous of the same degree, then $\frac{M(x,y)}{N(x,y)}$ is homogeneous of degree zero. For example $f(x,y)=\frac{x^2-y^2}{x^2+y^2}$ is homogeneous of degree zero.
2. The function $f(x,y)=x-5y+\sqrt{x^2+3y^2}$, is homogeneous of degree one, for

$$\begin{aligned} f(tx,ty) &= tx-5ty+\sqrt{(tx)^2+3(ty)^2} \\ &= t\left[x-5y+\sqrt{x^2+ty^2}\right]=tf(x,y) \end{aligned}$$

3. The function $F(x,y)=x^7\ln(x)-x^7\ln(y)$, is homogeneous of degree 7, because $f(x,y)=x\ln(x/y)$ and

$$f(tx,ty)=(tx)^7\ln(tx/ty)=t^7[x\ln(x/y)]=t^7f(x,y).$$

4. The functions

$$f(x,y) = x^2 + y^2 + \frac{x+y}{x-y} \quad \text{and } g(x,y) = 3x - 2y + e^{x-y},$$

are not homogeneous.

General Method

A first order differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ which can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F\left(\frac{y}{x}\right)$$

is called a **homogeneous differential equation**.

To solve the homogeneous differential equation:

by letting $u = \frac{y}{x}$, that is let $y = xu \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}u}{\mathrm{d}x} + u$, the equation then becomes

$$x\frac{\mathrm{d}u}{\mathrm{d}x} + u = F(u).$$

Hence

$$x\frac{\mathrm{d}u}{\mathrm{d}x} = F(u) - u.$$

This equation is clearly separable, and can be solved as such.

Or

by letting

$$u = \frac{x}{y}; y \neq 0,$$

that is let $x = yu \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} = y\frac{\mathrm{d}u}{\mathrm{d}y} + u,$

the equation then becomes

$$y\frac{\mathrm{d}u}{\mathrm{d}y} + u = F(u).$$

Hence

$$y\frac{\mathrm{d}u}{\mathrm{d}y} = F(u) - u.$$

This equation also is clearly separable, and can be solved as such.

Example

Solve the following differential equations:

1. $(x^2 - xy + y^2)dx - xydy = 0.$
2. $\frac{dy}{dx} + \frac{3xy + y^2}{x^2 + xy} = 0;\ x \neq 0$ and $y \neq -x.$
3. $ydx + x(\ln\left(\frac{x}{y}\right) - 1)dy = 0,\ y(1) = e.$
4. $x\frac{dy}{dx} - y = \sqrt{x^2 + y^2};\ x > 0.$

Solution

1. $(x^2 - xy + y^2)dx - xydy = 0.$

Solution The coefficients in this equation are both homogeneous and degree two in x and y. Let $u = \frac{y}{x}; x \neq 0$. Thus

$$y = ux \Rightarrow dy = udx + xdu$$

So the equation becomes

$$(x^2 - x(xu) + (xu)^2)dx - x(xu)(udx + xdu) = 0$$
$$(x^2 - x^2u + x^2u^2)dx - x^2u(udx + xdu) = 0$$
$$x^2(1 - u + u^2)dx - x^2u(udx + xdu) = 0$$

by dividing this equation by x^2 we obtain

$$(1 - u + u^2)dx - u(udx + xdu) = 0$$
$$(1 - u + u^2)dx - u^2dx - xudu = 0$$
$$(1 - u + u^2 - u^2)dx - xudu = 0$$

$$(1-u)dx - xudu = 0$$

hence we separate variables to get

$$\frac{dx}{x} - \frac{u}{1-u}du = 0; u \neq 1$$

$$\frac{dx}{x} + \frac{u}{u-1}du = 0$$

$$\frac{dx}{x} + \left[1 + \frac{1}{u-1}\right]du = 0$$

a family of solutions is seen to be

$$\ln x + u + \ln(u-1) = \ln c; c \neq 0$$

$$\ln x(u-1) + u = \ln c$$

$$x(u-1)e^u = c$$

$$x\left(\frac{y}{x} - 1\right)e^{\frac{y}{x}} = c.$$

2.

$$\frac{dy}{dx} + \frac{3xy + y^2}{x^2 + xy} = 0 \; ; \quad x \neq 0 \quad \text{and} \quad y \neq -x.$$

Solution The Coefficients of the differential equation are homogeneous, and it can be written in the form

$$(x^2 + xy)dy + \left(3xy + y^2\right) dx = 0 \; .$$

Let

$$u = \frac{x}{y}; y \neq 0,$$

hence

$$x = yu \implies dx = ydu + udy.$$

Then

$$(u^2y^2 + y^2u)dy + (3y^2u + y^2)(ydu + udy) = 0 \ ,$$

or

$$2y^2udy + 4y^2u^2dy = -y^3(3u + 1)du \ ,$$

$$-\frac{dy}{y} = \frac{3u + 1}{2u(2u + 1)}du \quad ; y \neq 0 \ , \quad u \neq 0 \quad \text{and} \quad u \neq \frac{-1}{2} \ ,$$

but

$$\frac{3u + 1}{2u(2u + 1)} = \frac{1}{2u} + \frac{1}{2(2u + 1)} \ ,$$

then

$$\ln|y| + \frac{1}{2}\ln|u| + \frac{1}{4}\ln|2u+1| = \ln|c| \quad ; c \neq 0$$

or

$$\ln\left[y^4u^2|2u+1|\right] = \ln c^4,$$

$$\ln\left[y^2x^2\left|\frac{2x}{y}+1\right|\right] = \ln c^4,$$

$$x^2|2xy+y^2| = c^4,$$

hence

$$yx^2(2x+y) = c_1.$$

is the family of curves defines the solutions of the DE, where $c_1 = c^4$ is an arbitrary constant.

3.

$$ydx + x\left(\ln\frac{x}{y} - 1\right)dy = 0 \quad , \quad y(1) = e$$

Solution The Coefficients of the differential equation are homogeneous with degree one .So we can put $u = \frac{x}{y}$ then

$$x = yu \implies dx = ydu + udy \ ,$$

we can suppose that $y > 0$ because the initial condition $y(1) > 0$. We obtain

$$y(ydu + udy) + yu(\ln u - 1)dy = 0,$$

$$y^2du + yu\ln u\ dy = 0 \ ,$$

hence

$$\frac{du}{u\ln u} + \frac{dy}{y} = 0 \quad ; \quad u \neq 1,$$

$$\ln|y\ln u| = c \implies |y\ln u| = e^c,$$

or

$$y\ln\left|\frac{x}{y}\right| = \mp e^c = c_1,$$

is the solution of differential equation. Now we use the initial condition $x = 1, y = e \implies c_1 = -e$, then the solution of the IVP for the DE is given by

$$y\ln\left(\frac{x}{y}\right) = -e \text{ , where } x > 0 \text{ and } y > 0$$

4.

$$x\frac{dy}{dx} - y = \sqrt{x^2 + y^2} \quad ; \quad x > 0 \quad .$$

Solution The differential equation is also homogeneous . Let $u = \frac{y}{x}$ then

$$y = ux \Longrightarrow \frac{dy}{dx} = u + xu',$$

hence

$$u + xu' - u = \sqrt{1 + u^2} \quad ,$$

or

$$\frac{du}{\sqrt{1 + u^2}} = \frac{dx}{x} \quad \Longrightarrow \quad \sinh^{-1}(u) - \ln x = c \ ,$$

So the solution of the DE is given by

$$\sinh^{-1}(\frac{y}{x}) - \ln x = c, \text{ where } c \text{ is an arbitrary constant .}$$

Summary

Let us summarize the steps to follow

1. Recognize that your equation is an homogeneous equation; that is, you need to check that $f(tx, ty) = f(x, y)$, meaning that f(tx,ty) is independent of the variable t;
2. Write out the substitution $u = y/x$;
3. Through easy differentiation, find the new equation satisfied by the new function u. You may want to remember the form of the new equation:
$$x\frac{\mathrm{d}u}{\mathrm{d}x} = F(u) - u \text{ or } y\frac{\mathrm{d}u}{\mathrm{d}y} = F(u) - u$$
4. Solve the new equation (which is always separable) to find u;

5. Go back to the old function y through the substitution $y = xu$;
6. If you have an **IVP**, use the initial condition to find the particular solution.

Exercise

Solve the following differential equations:

1. $(x^2 + y^2)dx - 2xydy = 0.$
2. $(x - y)dx + (2x + y)dy = 0.$
3. $2x^2y' - y(2x + y) = 0.$
4. $xdx + \sin^2\left(\frac{x}{y}\right)[ydx - xdy] = 0.$

Homogeneous Equations Requiring a Change of Variables

Solving Some Differential Equations by Using Appropriate Substitution

If we have a differential equation of the form

$$\frac{dy}{dx} = f(ax + by),$$

we use the substitution $u = ax + by$, then we get

$$\frac{du}{dx} = a + b\frac{dy}{dx}.$$

Example

Solve the following differential equations by using appropriate substitution:

1. $\frac{dy}{dx} = (-2x + y)^2 - 7, \quad y(0) = 0.$
2. $\frac{dy}{dx} = \frac{1-4x-4y}{x+y}; \; y \neq -x$
3. $\frac{dy}{dx} = \frac{x-y-3}{x+y-1}; \quad x + y - 1 \neq 0.$
4. $\frac{dy}{dx} = \frac{y(1+xy)}{x(1-xy)}; x > 0, y > 0, xy \neq 1.$ (Use the substitution $u = xy$)

Solution

1. $\frac{dy}{dx} = (-2x + y)^2 - 7, \quad y(0) = 0.$

If we let $u = -2x + y$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = -2 + \frac{\mathrm{d}y}{\mathrm{d}x}$, so the equation is transformed into

$$\frac{\mathrm{d}u}{\mathrm{d}x} + 2 = u^2 - 7 \quad \text{or} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = u^2 - 9.$$

The last equation is separable, thus

$$\frac{du}{u^2 - 9} = dx.$$

Using partial fractions

$$\frac{du}{(u-3)(u+3)} = dx \quad \text{or} \quad \frac{1}{6}\left[\frac{1}{u-3} - \frac{1}{u+3}\right]$$

and then integrating yields

$$\frac{1}{6}\left[\ln\frac{u-3}{u+3}\right] = x + c_1 \quad \text{or} \quad \frac{u-3}{u+3} = e^{6x+6c_1} = ce^{6x}.$$

Finally, applying the initial condition $y(0) = 0$ to get the particular solution

$$y = 2x + \frac{3(1 - e^{6x})}{(1 + e^{6x})}.$$

2. $\frac{dy}{dx} = \frac{1-4x-4y}{x+y}$; $y + x \neq 0$.

We see that the two straight lines $1 - 4x - 4y$ and $x + y$ are parallels, i.e if we have equation in form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

and

$$\frac{a_1}{b_1} = \frac{a_2}{b_2}.$$

The figure below shows the nature of the two lines

$$a_1x + b_1y + c_1$$

$$a_2x + b_2y + c_2$$

In this case we let $u = x + y$. Hence

$$y' = u' - 1,$$

and we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-4u}{u} = \frac{\mathrm{d}u}{\mathrm{d}x} - 1 \quad \text{or} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1-3u}{u}$$

The last equation is separable, thus

$$\frac{u}{1-3u}du = dx.$$

and then integrating yields

$$\frac{x+y}{3} + \frac{1}{9}\ln|1-3u| + x = c.$$

3. $\frac{dy}{dx} = \frac{x-y-3}{x+y-1}; \quad x + y - 1 \neq 0.$

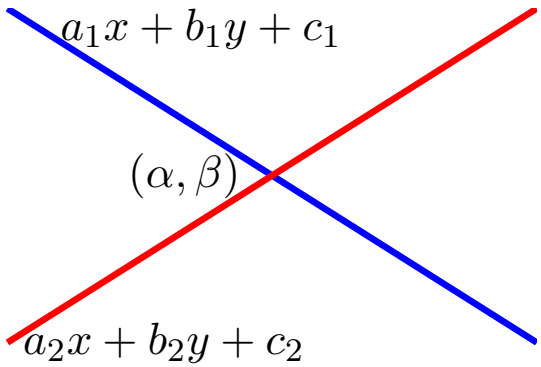
We see that the two straight lines $x - y - 3$ and $x + y - 1$ are not parallels, i.e if we have equation in form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

and

$$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}.$$

The figure below shows the nature of the two lines



In this case we need first to find the **intersection point** (α, β), then we use the substitutions

$$x = u + \alpha \quad \text{and} \quad y = v + \beta$$

Thus, in this example we need to solve the two equations to find the **intersection point** which is $(2, -1)$.

Now we will use the substitutions

$$x = u + 2 \quad \text{and} \quad y = v - 1$$

thus,

$$dx = du \quad \text{and} \quad dy = dv.$$

Then

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{u + 2 - (v - 1) - 3}{u + 2 + (v - 1) - 1} = \frac{u - v}{u + v}.$$

So now we have this homogeneous differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{u-v}{u+v},$$

so we let $t = \frac{v}{u}$, where $u \neq 0$. Then $v = ut$ and

$$\frac{\mathrm{d}v}{\mathrm{d}u} = t + u\frac{\mathrm{d}t}{\mathrm{d}u},$$

thus,

$$t + u\frac{\mathrm{d}t}{\mathrm{d}u} = \frac{u-ut}{u+ut} \quad \text{or} \quad u\frac{\mathrm{d}t}{\mathrm{d}u} = \frac{1-t}{1+t} - t = \frac{1-2t-t^2}{1+t}$$

by integrating

$$\int \frac{du}{u} = \int \frac{1+t}{1-2t-t^2}dt$$

$$\int \frac{du}{u} = -\frac{1}{2}\int \frac{-2-2t}{1-2t-t^2}dt$$

$$\ln u + \frac{1}{2}\ln\left(1-2t-t^2\right) = c$$

$$\ln\left[u^2(1-2\frac{v}{u}-\frac{v^2}{u^2}\right] = 2c$$

$$\ln\left[u^2-2vu-v^2\right] = 2c$$

$$e^{\ln\left[u^2-2vu-v^2\right]} = e^{2c}$$

$$u^2-2vu-v^2 = c_1$$

thus, the solution is $(x-2)^2-2(x-2)(y+1)-(y+1)^2 = c_1$.

Exact Differential Equations

A differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0,$$

is called **exact**, if there is a function F of x and y such that

$$dF(x,y) = M(x,y)dx + N(x,y)dy = 0.$$

Recall that the total differential of a function $F(x,y)$ is

$$dF(x,y) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy.$$

provided that the partial derivatives of the function F is exists.

Theorem (Criterion for an Exact Differential)

Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b, c < y < d$. Then a necessary and sufficient condition that $M(x,y)dx + N(x,y)dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Example

Prove that the following differential equations are exact and find their solutions

1. $(6x^2 + 4xy + y^2)dx + (2x^2 + 2xy - 3y^2)dy = 0$
2. $\left[\cos x \ln(2y - 8) + \frac{1}{x}\right] dx + \frac{\sin x}{y-4} dy$; $x \neq 0$ and $y > 4$.
3. $(e^{2y} - y\cos xy)dx + (2xe^{2y} - x\cos xy + 2y)dy = 0$

To prove that we need to check for the differential equation

$$M(x, y)dx + N(x, y)dy = 0,$$

if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

then the differential equation is exact.

Solution

1. $(6x^2 + 4xy + y^2)dx + (2x^2 + 2xy - 3y^2)dy = 0$

$M(x, y) = 6x^2 + 4xy + y^2 \Rightarrow \frac{\partial M}{\partial y} = 4x + 2y$ $N(x, y) = 2x^2 + 2xy - 3y^2$

$\Rightarrow \frac{\partial N}{\partial x} = 4x + 2y$ i.e. $\frac{\partial M}{\partial y} = 4x + 2y = \frac{\partial N}{\partial x}$. Thus, the differential

equation is exact.

Now to find the solution

$$\int (6x^2 + 4xy + y^2)dx = 2x^3 + 2x^2y + y^2x.$$

$$\int (2x^2 + 2xy - 3y^2)dy = 2x^2y + xy - y^3.$$

Thus, the family solution is

$$2x^3 + 2x^2y + y^2x - y^3 = c$$

.

2. $\left[\cos x \ln(2y-8) + \frac{1}{x}\right] dx + \frac{\sin x}{y-4} dy$; $x \neq 0$ and $y > 4$.

$$M(x,y) = \left[\cos x \ln(2y-8) + \frac{1}{x}\right] \Rightarrow \frac{\partial M}{\partial y} = 2\cos x \frac{1}{2y-8} = \cos x \frac{1}{y-4}$$
$$N(x,y) = \frac{\sin x}{y-4} \Rightarrow \frac{\partial N}{\partial x} = \cos x \frac{1}{y-4}$$

i.e. $\frac{\partial M}{\partial y} = \cos x \frac{1}{y-4} = \frac{\partial N}{\partial x}$.

Thus, the differential equation is exact.

Now to find the solution

$$\begin{aligned} \int \left[\cos x \ln(2y-8) + \frac{1}{x}\right] dx &= \sin x \ln(2y-8) + \ln x \\ &= \sin x \ln[2(y-4)] + \ln x \end{aligned}$$

$$\int \frac{\sin x}{y-4} dy = \sin x \ln(y-4),$$

Thus, the family solution is

$$\sin x \ln(y-4) + \ln x + c = 0$$

3. $(e^{2y} - y\cos xy)dx + (2xe^{2y} - x\cos xy + 2y)dy = 0$

$M(x,y) = e^{2y} - y\cos xy \Rightarrow \frac{\partial M}{\partial y} = 2e^{2y} - \cos xy + xy\sin xy$

$N(x,y) = 2xe^{2y} - x\cos xy + 2y \Rightarrow \frac{\partial N}{\partial x} = 2e^{2y} - \cos xy + xy\sin xy$

i.e. $\frac{\partial M}{\partial y} = 2e^{2y} - \cos xy + xy\sin xy = \frac{\partial N}{\partial x}$.

Thus, the differential equation is exact.

Now to find the solution

$$\int (e^{2y} - y\cos xy) \ \ dx = xe^{2y} - \sin xy$$

$$\int (2xe^{2y} - x\cos xy + 2y) \ \ dy = xe^{2y} - \sin xy + y^2,$$

Thus, the family solution is

$$xe^{2y} - \sin xy + y^2 + c = 0$$

.

Integrating Factor

Consider a first order differential equation

$$M(x,y)dx + N(x,y)dy = 0, \tag{7}$$

where M, N and $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous on a certain region R in xy-plane. Suppose that the equation (7) is **not exact**, i.e

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Definition

A function $\mu(x,y)$ is called an **integrating factor** of (7) if the differential equation

$$(\mu M)dx + (\mu N)dy = 0, \tag{8}$$

is **exact**, i.e

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}. \tag{9}$$

In other words, if the equation (7) is **not exact**, we can often make it so by multiplying throughout by an **integrating factor** $\mu(x, y)$ and the finding $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$. The integrating factors are able to be determined by solving

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

for μ.

The integrating factor will be in one of the following forms

1. $\mu = \mu(x)$
2. $\mu = \mu(y)$
3. $\mu = \mu(x, y) = x^m y^n$

We can rewrite the equation (9) as follows:

$$N_{\mu_x} - M_{\mu_y} = (M_y - N_x)_\mu \tag{10}$$

In general, it is very difficult to solve the equation (10). In this section we will only consider that μ is a one variable function (x or y, not both). There are two cases:

1. If μ depends on x ($\mu = \mu(x)$). Then $\mu_y = 0$, so the equation (10) becomes
$$\frac{1}{\mu}\mu_x = \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{M_y - N_x}{N},$$
so
$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}.$$
2. If μ depends on y ($\mu = \mu(y)$). Then $\mu_x = 0$, so the equation (10) becomes
$$\frac{1}{\mu}\mu_y = \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}y} = \frac{N_x - M_y}{M},$$
so
$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}.$$

We summarize that for the differential equation

$$M(x,y)dx + N(x,y)dy = 0,$$

as following

1. If $(M_y - N_x)/N$ is a function of x only, then the integrating factor for the differential equation is

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}.$$

2. If $(N_x - m_y)/M$ is a function of y only, then the integrating factor for the differential equation is

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}.$$

Example

Solve the following differential equations:

1. $xydx + (2x^2 + 3y^2 - 20)dy = 0; x \neq 0, y > 0.$
2. $(4xy + 3y^2 - x)dx + x(x + 2y)dy = 0, x(x + 2y) \neq 0.$

Solution

1. $xydx + (2x^2 + 3y^2 - 20)dy = 0; x \neq 0, y > 0.$

$M(x,y) = xy \Rightarrow \frac{\partial M}{\partial y} = x$

$N(x,y) = 2x^2 + 3y^2 - 20 \Rightarrow \frac{\partial N}{\partial x} = 4x$ so, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Thus, the differential equation is not exact.

Now let's us find the solution

$$\frac{M_y - N_x}{N} = \frac{4x - x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20},$$

we note that the quotient is depended on x and y.

So we need to find

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3}{y} = g(y),$$

we note that the quotient is depended only on y, thus the integrating factor

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy} = e^{\int g(y) dy} = e^{\int \frac{3}{y} dy} = e^{3 \ln y} = e^{\ln y^3} = y^3.$$

Then we multiply the equation by $\mu(y) = y^3$,
thus, the equation becomes

$$xy^4 dx + (2x^2y^3 + 3y^5 - 20y^3) dy = 0$$

this equation is exact now, that is because

$$\frac{\partial M}{\partial y} = 4xy^3 = \frac{\partial N}{\partial x}.$$

So

$$\int xy^4 \, dx = \frac{1}{2}x^2y^4$$

$$\int (2x^2y^3 + 3y^5 - 20y^3) \, dy = \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4,$$

Thus, the family solution is

$$\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 + c = 0$$

.

2. $(4xy + 3y^2 - x)dx + x(x + 2y)dy = 0, x(x + 2y) \neq 0.$

$M(x, y) = 4xy + 3y^2 - x \Rightarrow \frac{\partial M}{\partial y} = 4x + 6y$

$N(x, y) = x(x + 2y) \Rightarrow \frac{\partial N}{\partial x} = 2x + 2y,$

so, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Thus, the differential equation is not exact.

Now let's us find the solution

$$\frac{M_y - N_x}{N} = \frac{4x + 6y - 2x - 2y}{x(x + 2y)} = \frac{2(x + 2y)}{x(x + 2y)} = \frac{2}{x} = f(x),$$

we note that the quotient is depended on x, thus the integrating factor

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx} = e^{\int f(x)dx} = e^{\int \frac{2}{x} dy} = e^{2 \ln x} = e^{\ln x^2} = x^2.$$

Then we multiply the equation by $\mu(x) = x^2$,
thus, the equation becomes

$$(4x^3y + 3x^2y^2 - x^3)dx + (x^4 + 2x^3y)dy = 0$$

this equation is exact now, that is because

$$\frac{\partial M}{\partial y} = 4x^3 + 6x^3y = \frac{\partial N}{\partial x}.$$

So

$$\int (4x^3y + 3x^2y^2 - x^3)\ dx = x^4y + x^3y^2 - \frac{1}{4}x^4,$$

$$\int (x^4 + 2x^3y)\ dy = x^4y + x^3y^2.$$

Thus, the family solution is

$$x^4y + x^3y^2 - \frac{1}{4}x^4 + c = 0$$

Example

Find m, n such that

$$\mu(x,y) = x^m y^n,$$

is an integrating factor of the differential equation

$$(2y^2 + 4x^2y)dx + (4xy + 3x^3)dy = 0.$$

Solution

$$(2y^2 + 4x^2y)dx + (4xy + 3x^3)dy = 0,$$

we need to find m and n such that the equation

$$(2x^m y^{n+2} + 4x^{m+2}y^{n+1})dx + (4x^{m+1}y^{n+1} + 3x^{m+3}y^n)dy = 0,$$

thus,

$$\frac{\partial M}{\partial y} = 2(n+2)x^m y^{n+1} + 4(n+1)x^{m+2}y^n,$$

$$\frac{\partial N}{\partial x} = 4(m+1)x^m y^{n+1} + 3(m+3)x^{m+2}y^n.$$

For the exactness we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

by equating coefficients we have that

$$2(n+2) = 4(m+1) \Rightarrow 2n - 4m = 0 \Rightarrow n = 2m,$$

and

$$4(n+1) = 3(m+3) \Rightarrow 4n - 3m - 5 = 0.$$

Therefor,

$$m = 1 \quad \text{and} \quad n = 2.$$

Thus, integrating factor of the differential equation

$$\mu(x, y) = xy^2.$$

Therefor the solution for the given differential equation is

$$x^2y^4 + x^4y^3 = c.$$

Exercises

Solve the following differential equations:

1. $(x^2 + y^2 + 1)dx + x(x - 2y)dy = 0.$
2. $y(x + y + 1)dx + x(x + 3y + 2)dy = 0; y(x + y + 1) \neq 0$

The General Solution of a Linear Differential Equations

Consider the linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{11}$$

where P and Q are continuous function on the interval (a, b).
The integrating factor of the differential equation (11) is

$$\mu(x) = e^{\int P(x)dx}.$$

The general solution of equation (11) is given by

$$y\mu(x) = \int \mu(x)Q(x)dx + C.$$

Since $\mu(x) \neq 0$, for $x \in (a, b)$, then we can write

$$y\mu(x) = \int \mu(x)Q(x)dx + C,$$

$$y(x) = e^{-\int P(x)dx} \int \mu(x)Q(x)dx + Ce^{-\int P(x)dx}.$$

Example

Solve the following differential equations:

1. $x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = x^3$.
2. $(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + xy + x^3 + x = 0$.
3. $(y - x + xy\cot x)dx + xy + xdy = 0; 0 < y < \pi$ with initial value problem $y(\pi/2) = 0$.

Solution

1. $x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = x^3$.

The equation can be written in the form $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2y}{x} = x^2$, we can see that the equation is in a Linear Differential Equation Form. Where $P(x) = \frac{2}{x}$, and $Q(x) = x^2$.

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{2}{x}dx} = x^2.$$

The general solution will be in form

$$y\mu(x) = \int \mu(x)Q(x)dx + C,$$

so,

$$yx^2 = \int x^2x^2dx \Rightarrow yx^2 = \int x^4dx.$$

Thus, the general solution is

$$yx^2 = \frac{1}{5}x^5 + c.$$

2. $(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + xy + x^3 + x = 0.$

The equation can be written in the form $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x}{1+x^2}y = -x$, we can see that the equation is in a Linear Differential Equation Form. Where $P(x) = \frac{x}{1+x^2}$, and $Q(x) = -x$.

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{x}{1+x^2}dx} = e^{\frac{1}{2}\ln\left(1+x^2\right)} = (1+x^2)^{\frac{1}{2}}.$$

The general solution will be in form

$$y\mu(x) = \int \mu(x)Q(x)dx + C,$$

so,

$$y\sqrt{1+x^2} = -\int x\sqrt{1+x^2}dx,$$

thus, the general solution is

$$y\sqrt{1+x^2} = \frac{-1}{3}(1+x^2)^{\frac{3}{2}} + c.$$

Exercise

Find the initial value problem **(IVP)**

$$(y - x + xy\cot x)dx + xdy = 0; 0 < y < \pi$$

and

$$y(\pi/2) = 0.$$

(Hint: $P(x) = \frac{1-x\cot x}{x}$ and $Q(x) = 1$)

Bernoulli's Equation

The Bernoulli's equation is a first order differential equation, which can be written in the form

$$y' + P(x)y = Q(x)y^n, \tag{12}$$

where $n \in \mathbb{R}$.

1. If $n = 0$ then the equation (12) is a linear first order differential equation and we can solve it as we saw before.
2. If $n = 1$ then the equation (12) is becomes a differential equation with separable variables, and we can solve it by by separating the variables.

3. If $n \neq 0$ and $n \neq 1$ then the equation (12) can be written in the form

$$y^{-n}y' + P(x)y^{-n+1} = Q(x).$$

Now we let $u = y^{-n+1}$, then we have

$$\frac{\mathrm{d}u}{\mathrm{d}x} = (-n+1)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}$$

or

$$u' = (-n+1)y^{-n}y'.$$

$$\frac{1}{-n+1}u' + P(x)u = Q(x)$$

or

$$u' + (-n+1)P(x)u = (-n+1)Q(x),$$

which is a linear first order differential equation and we can solve it.

Bernoulli's Equation

Example

Solve the following differential equations:

1. $\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = xe^{-x^2}y^3$.
2. $y(6y^2 - x - 1)dx + 2xdy = 0; x \neq 0$.

Solution

1. $\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = xe^{-x^2}y^3$.

We can see that; the equation is in the Bernoulli's Equation Form. The equation can be written in the form

$$y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy^{-2} = xe^{-x^2}.$$

Now we let $u = y^{-2}$, thus we have

$$u' = -2y^{-3}y'.$$

Thus, the equation becomes

$$\frac{-1}{2}\frac{\mathrm{d}u}{\mathrm{d}x} + 2xu = xe^{-x^2}.$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} - 4xu = -2xe^{-x^2}, \tag{13}$$

thus, the equation (13) is in linear first order differential equation and we can solve it. Where $P(x) = -4x$, and $Q(x) = -2xe^{-x^2}$.

$$\mu(x) = e^{\int P(x)dx} = e^{\int -4x\ dx} = e^{-2x^2}.$$

The general solution will be in form

$$u\mu(x) = \int \mu(x)Q(x)dx + C,$$

so,

$$ue^{-2x^2} = \int e^{-2x^2}(-2xe^{-x^2})\ dx$$

$$ue^{-2x^2} = \frac{-2}{-6}\int -6xe^{-3x^2}\ dx$$

$$ue^{-2x^2} = \frac{1}{3}e^{-3x^2} + c,$$

$$u = \frac{1}{3}e^{-x^2} + ce^{-2x^2},$$

thus, the general solution is

$$\frac{1}{y^2} = \frac{1}{3}e^{-x^2} + ce^{-2x^2}.$$

2. $y(6y^2 - x - 1)dx + 2xdy = 0; x \neq 0.$

The equation can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{x+1}{2x}y = \frac{-3}{x}y^3.$$

So we have the Bernoulli's Equation, and it might be written in the form

$$y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{x+1}{2x}y^{-2} = \frac{-3}{x}.$$

Now we let $u = y^{-2}$, thus we have

$$u' = -2y^{-3}y'.$$

Thus, this equation becomes

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{x+1}{2x}u = \frac{6}{x}. \tag{14}$$

Thus, the equation (14) is in linear first order differential equation and we can solve it. Where $P(x) = \frac{x+1}{2x}$, and $Q(x) = \frac{6}{x}$.

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{x+1}{2x}\ dx} = xe^x.$$

The general solution will be in form

$$u\mu(x) = \int \mu(x)Q(x)dx + C,$$

$$uxe^x = 6e^x + C$$

thus, the general solution is

$$y^2(6 + Ce^{-x}) = x.$$

Exercises

Solve the following differential equations:

1. $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x}y = -2e^x y^2$.
2. $(2y^3 - x^3)dx + 2xy^2 dy = 0; x \neq 0$ with **IV** $y(1) = 1$.

Applications of First Order Differential Equation

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Applications of First Order Differential Equation

- Orthogonal Trajectories
  - How to Find Orthogonal Trajectories
- Growth and Decay
- Newton's Law of cooling

Suppose that we have a family of curves given by

$$F(x, y, c) = 0, \tag{1}$$

and another family of curves given by

$$G(x, y, k) = 0, \tag{2}$$

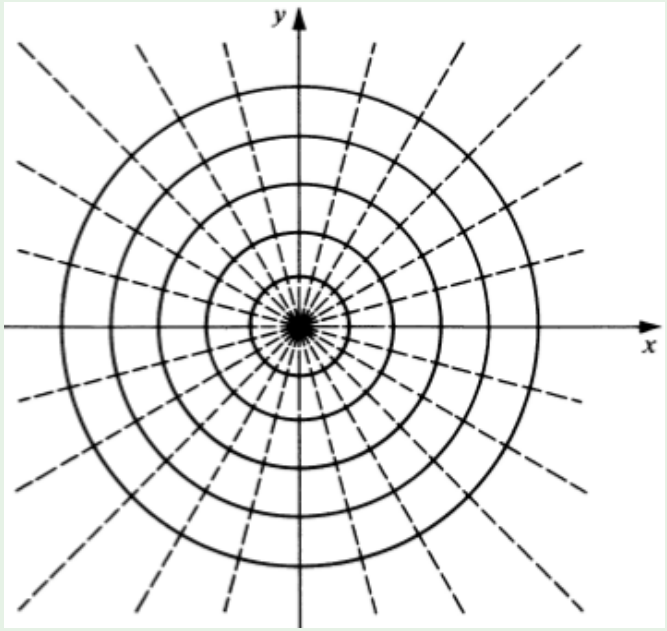
such that at any intersection of a curve of the family $F(x, y, c)$ with a curve of the family $G(x, y, k) = 0$, the tangents of the curves are perpendicular.

Therefore, are two families of curves that always intersect perpendicularly.

Orthogonal trajectories,

Example

The family of circles represented by $x^2 + y^2 = c$, with center at the origin, and the family $y = kx$ of straight lines through the origin, are **orthogonal trajectories** of each other, as shown in the figure.



How to Find Orthogonal Trajectories

To find the orthogonal trajectories of the family

$$F(x, y, c) = 0, \tag{3}$$

Step1: Differentiate (3) implicitly with respect to x to get a relation of the form (3)

$$g\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, c\right); \tag{4}$$

Step2: Eliminate the parameter c from (3), and (4) to obtain the differential equation

$$F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}\right) \tag{5}$$

corresponding to the first family (3);
Step3: Replace $\frac{\mathrm{d}y}{\mathrm{d}x}$ by $\frac{-1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$ in (5) to obtain the differential equation

$$H\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}\right) \tag{6}$$

of the orthogonal trajectories (as shown in the figure below);

Step4: General solution of (6) gives the required orthogonal trajectories.

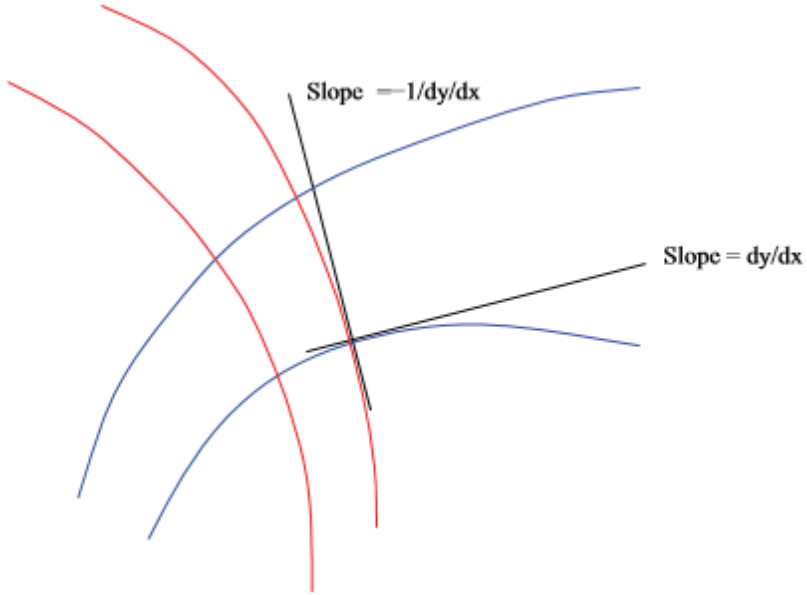


Figure: Orthogonal trajectories

Example (1)

Find the orthogonal trajectories of family of straight lines through the origin.

Solution: The family of straight lines through the origin is given by

$$y = kx, \tag{7}$$

To find the orthogonal trajectories, we follow the previous four steps:

Step1: Differentiate (7) implicitly with respect to x, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = k, \tag{8}$$

Step2: Eliminate the parameter k from (7), and (8), we obtain the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x}, \tag{9}$$

This gives the differential equation of the family (7).

Step3: Replacing $\frac{\mathrm{d}y}{\mathrm{d}x}$ by $\frac{-1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$ in (9) we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}, \tag{10}$$

Step4: Solving differential equation (10), we obtain

$$x^2 + y^2 = c. \tag{11}$$

Thus, the orthogonal trajectories of family of straight lines through the origin is given by (11). Note that (11) is the family of circles with centre at the origin.

Example (2)

Find the orthogonal trajectories of the family

$$cx^2 - y^2 = 1, \tag{12}$$

Solution: To find the orthogonal trajectories, we follow the previous four steps:

Step1: Differentiate (12) implicitly with respect to x, we obtain

$$2cx - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{13}$$

Step2: Eliminate the parameter c. From (12) we have

$$c = \frac{1 + y^2}{x^2}$$

Thus, we obtain the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + y^2}{xy}, \tag{14}$$

This gives the differential equation of the family (12).

Step3: Replacing $\frac{\mathrm{d}y}{\mathrm{d}x}$ by $\frac{-1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$ in (14) we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-xy}{1+y^2}. \tag{15}$$

Step4: Solving differential equation (15) by method of separation of variables, we obtain

$$\int \frac{1+y^2}{y}\, dy = -\int x\, dx$$

$$\int \left(\frac{1}{y} + y\right)\, dy = -\int x\, dx$$

$$\ln y + (y^2/2) = (-x^2/2) + c_1$$

$$2\ln y + y^2 + x^2 = c_1. \tag{16}$$

Thus, the required equation of orthogonal trajectories is given by (16).

Example (3)

Find the orthogonal trajectories of the family

$$y^2 = cx^3, \tag{17}$$

Solution: To find the orthogonal trajectories, we follow the previous four steps:

Step1: Differentiate (17) implicitly with respect to x, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3cx^2}{2y}. \tag{18}$$

Step2: Eliminate the parameter c. From (17) we have

$$c = \frac{y^2}{x^3}$$

Thus, we obtain the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y}{2x} = f(x, y), \tag{19}$$

This gives the differential equation of the family (17).

Step3: Replacing $\frac{\mathrm{d}y}{\mathrm{d}x}$ by $\frac{-1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$ in (14) we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x}{3y}. \tag{20}$$

Step4: Solving differential equation (20) by method of separation of variables, we obtain

$$\int 3y\ dy = -\int 2x\ dx$$

$$\frac{3}{2}y^2 + x^2 = c_1. \tag{21}$$

Thus, the required equation of orthogonal trajectories is given by (21).

Example (4)

Find the orthogonal trajectories of the family

$$x^3 + 3xy^2 = c \tag{22}$$

Solution: To find the orthogonal trajectories, we follow the previous four steps:

Step1: Differentiate (22) implicitly with respect to x, we obtain

$$3x^2 + 3y^2 + 6xy\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{23}$$

Step2: Equation (23) has no parameter, so, we will go to the next step.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x^2 + y^2}{2xy} = f(x, y), \tag{24}$$

This gives the differential equation of the family (22).

Step3: Replacing $\frac{\mathrm{d}y}{\mathrm{d}x}$ by $\frac{-1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$ in (24) we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy}{x^2+y^2}. \tag{25}$$

This gives the differential equation of the orthogonal trajectories. Now we have to solve (25) to get the required equation of orthogonal trajectories.

Step4: Solving differential equation (25). We see that the equation (25) is a homogeneous differential equation

$$(x^2+y^2)dy - 2xydx = 0,$$

by substituting $x = vy \Rightarrow 1 = v\frac{\mathrm{d}y}{\mathrm{d}x} + y\frac{\mathrm{d}v}{\mathrm{d}x} \Rightarrow dx = vdy + ydv$.
After completing the solution of this homogeneous differential equation, we obtained the equation of family of orthogonal trajectories

$$(y^2 - x^2) = cy.$$

Exercises

1. Find the orthogonal trajectories of the circles

$$x^2 + (y - c)^2 = c^2, \quad c \neq 0$$

2. Find the orthogonal trajectories of the family of curves

$$y = x + ce^{-x}$$

3. Find the orthogonal trajectories of the family of curves

$$2x^2 + y^2 = 6cx$$

4. Find the orthogonal trajectories of the family of curves

$$2y + x + ce^{2y} = 0$$

Growth and Decay

In many natural phenomena, quantities grow or decay at a rate proportional to their size. For example, if $y = y(t)$ is the number of individuals in a population of animals or bacteria at time t, then it seems reasonable to expect that the rate of growth $y'(t)$ is proportional to the population $y(t)$; that is, $y'(t) = ky(t)$ for some constant k.

The mathematical model given by the equation $y'(t) = ky(t)$ can be predicted what actually happens fairly accurately under ideal conditions (unlimited environment, adequate nutrition, immunity to disease). Also, we can see many examples in nuclear physics, chemistry and finance.

In general, if y(t) is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size y(t) at any time, then

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky, \tag{26}$$

where k is a constant, and Equation (26) is sometimes called the **law of natural growth** (if $k > 0$) or the **law of natural decay** (if $k < 0$).
Thus, the law of Exponential Growth and Decay can be written as

$$y = ce^{kt},$$

c is the initial value and can be found from the initial condition $y(t_0) = y_0$
k is the constant of proportionality, which is can be found from an additional condition which might be given in the problem.

Note

If $k > 0$ the exponential growth occurs, and if $k < 0$ the exponential decay occurs.

To proof that let us take some initial time quantity is known and is $y(t_0) = y_0$. The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky$$

is separable differential equation and we can solve it.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky$$

$$\frac{dy}{y} = kdt$$

$$\int \frac{dy}{y} = \int kdt$$

$$\ln y = kt + c$$

$$e^{\ln y} = e^{kt+c}$$

$$y = e^c e^{kt}$$

$$y = c_1 e^{kt};\ c_1 = \pm e^c$$

Using the initial condition $y(0) = y_0$, i.e $t_0 = 0$, $y = y_0$

$$y_0 = c_1 e^0 \Rightarrow y_0 = c_1$$

$$y = y_0 e^{kt}.$$

To find the additional constant k we need additional condition which might be given in the problem.

Example (1)

A certain culture of bacteria grows at rate proportional to its size. If the size doubles in 4 days, find the time required for the culture to increase to 10 times to its original size.

Solution Let $p(t)$ be the size of the culture after t days.

$$\frac{\mathrm{d}p}{\mathrm{d}t} = kp$$

we will use the initial condition

$$p(0) = p_0$$

to find the arbitrary constant c, and we will find the additional constant k by using the additional condition

$$p(4) = 2p_0.$$

We have

$$p = ce^{kt}$$

from the initial condition $p(0) = p_0$ i.e. $t = 0$, $p = p_0$ we will have arbitrary constant c, thus

$$p_0 = ce^0 \Rightarrow c = p_0,$$

hence, we have

$$p = p_0 e^{kt}.$$

Now by using the the additional condition $p(4) = 2p_0$ i.e. $t = 4$, we we can find the additional constant k

$$2p_0 = p_0 e^{4k}$$

$$e^{4k} = 2$$

$$\ln\left(e^{4k}\right) = \ln 2$$

$$4k = \ln 2 \Rightarrow k = \frac{\ln 2}{4} \simeq 0.173$$

Thus, the time is required for the culture to increase 10 times to its original size can be found from

$$10p_0 = p_0 e^{0.173t}$$

$$e^{0.173t} = 10$$

$$\ln\left(e^{0.173t}\right) = \ln 10$$

$$t = \frac{\ln 10}{0.173}$$

$$t \simeq 13.31 \text{ days.}$$

Example (2)

Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.) What is the relative growth rate k? Use the model to estimate the world population in 1993 and to predict the population in the year 2020.

Solution We measure the population $p(t)$ in millions of people. We have

$$\frac{\mathrm{d}p}{\mathrm{d}t} = kp \;\Rightarrow\; p = ce^{kt}$$

and we have the initial condition

$$p(t_0) = p_0 \;\Rightarrow\; p(0) = 2560,$$

thus, we can find the arbitrary constant c

$$p = ce^{kt} \;\Rightarrow\; p(0) = ce^0 \;\Rightarrow\; 2560 = c.$$

Now we will find the additional constant k (the relative growth rate) by using the additional condition

$$p(10) = 3040.$$

$$p = ce^{kt}$$

$$3040 = 2560e^{10k}$$

$$e^{10k} = \frac{3040}{2560}$$

$$\ln\left(e^{10k}\right) = \ln 1.1875$$

$$10k = \ln 1.1875 \;\Rightarrow\; k = \frac{\ln 1.1875}{10} \simeq 0.01785.$$

The relative growth rate is about 1.7% per year and the model is

$$p(t) = 2560e^{0.017185t}.$$

We estimate that the world population in 1993 was by using the model

$$p(t) = 2560e^{0.017185t}.$$

$$p(43) = 2560e^{0.017185(43)} \simeq 5360 \text{ million}.$$

The model predicts that the population in 2020 will be

$$p(70) = 2560e^{0.017185(70)} \simeq 8524 \text{ million}.$$

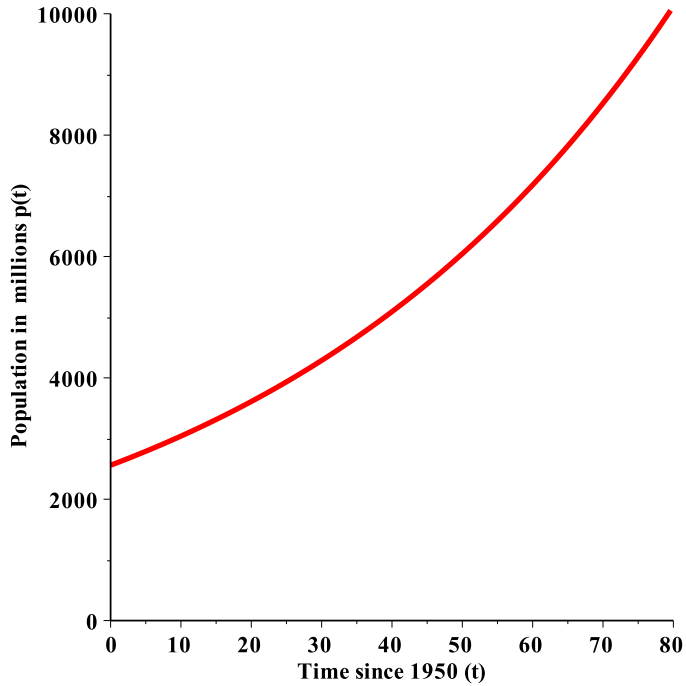


Figure: A model for world population growth in the 2nd half of the 20th century

Exercise

The population of a town at a rate proportional to the population size at any time. Its initial population of 1000 increases by 10% in 5 years. What will be the population after 50 years?

(Hint: $p(0) = 1000$, $p(5) = 1000 + (1000/10) = 1100$)

Example (3)

A radio active material has an initial mass 100mg. After two years it is left to 75mg. Find the amount of the material at any time. What is the period of its half-life?

Solution We measure the amount of the material present at any time t. We have

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky \;\Rightarrow\; y = ce^{kt}$$

and we have the initial condition

$$y(t_0) = y_0 \;\Rightarrow\; y(0) = 100,$$

thus, we can find the arbitrary constant c

$$y = ce^{kt} \;\Rightarrow\; y(0) = ce^0 \;\Rightarrow\; 100 = c.$$

$$y = 100e^{kt}$$

Now we will find the additional constant k by using the additional condition

$$y(2) = 75.$$

$$75 = 100e^{2k}$$

$$e^{2k} = \frac{75}{100}$$

$$\ln\left(e^{10k}\right) = \ln 0.75$$

$$2k = \ln 0.75 \;\Rightarrow\; k = \frac{\ln 0.75}{2} \simeq -0.1438.$$

Thus, we have

$$y(t) = 100e^{-0.1438t}.$$

$$y(t) = 100e^{-0.1438t},$$

from the latest equation we will find the Half-life of the material which is the time when $y = 50$mg.

$$50 = 100e^{-0.1438t}$$

$$e^{-0.1438t} = 0.5$$

$$\ln e^{-0.1438t} = \ln 0.5$$

$$-0.1438t = \ln 0.5$$

$$t = \frac{\ln 0.5}{-0.1438} \simeq 4.82 \text{ years.}$$

Exercise

Initially there were 100 milligrams (mg.) of a radioactive substance present after 6 hours the mass decreased by 3%. If the rate of decay is proportional to the amount remaining after 24 hours. Determine the half-life of radioactive substance.

Newton's Law of cooling

Newtons Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large.

If we let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings, then we can formulate Newtons Law of Cooling as a differential equation:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_s), \tag{27}$$

where k is a constant of proportionality.

We could solve equation (27) as a separable differential equation

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_s)$$

$$\frac{dT}{T - T_s} = kdt$$

$$\int \frac{dT}{T - T_s} = \int kdt$$

$$\ln(T - T_s) = kt + c_1$$

$$e^{\ln(T - T_s)} = e^{kt + c_1}$$

$$T - T_s = e^{kt}e^{c_1} = ce^{kt},$$

thus,

$$T = T_s + ce^{kt}. \tag{28}$$

The constant of integration c can be calculated by using the initial condition and the constant of proportionality k can be calculated by using an additional condition.

Example (1)

A glass of a hot water has an initial temperature $80°$C, placed in a room where the temperature is $30°$C. After one minute the water temperature drops to $70°$C. What will be the temperature after 3 minutes? At what time the water cools down to $40°$C?

Solution We have the Newtons Law of Cooling is given from

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_s) \;\Rightarrow\; T = T_s + ce^{kt}.$$

Also we have the initial condition $T(0) = 80°$, the temperature of the surrounding $T_s = 30°$, and the additional condition $T(1) = 70°$

$$T = T_s + ce^{kt} \;\Rightarrow\; T(0) = 30 + ce^0 \;\Rightarrow\; 80 = 30 + c \;\Rightarrow\; c = 50.$$

Thus,

$$T = 30 + 50e^{kt}.$$

Now we will find the additional constant k by using the additional condition $T(1) = 70°$

$$T = 30 + 50e^{kt},$$

$$T(1) = 30 + 50e^{k} \;\Rightarrow\; 70 = 30 + 50e^{k} \;\Rightarrow\; e^{k} = \frac{40}{50}$$

$$\ln\left(e^{k}\right) = \ln 0.8$$

$$k = \ln(0.8).$$

Thus, the temperature of the water at any time is given by

$$T(t) = 30 + 50e^{\ln(0.8)t},$$

so, when $t = 3$ we have the temperature of the water

$$T(3) = 30 + 50e^{3\ln(0.8)}$$

$$T(3) = 30 + 25.6 = 55.6°\mathrm{C}.$$

Now we will find the time t when the water cools down to $40°$

$$T = 30 + 50e^{kt},$$

$$40 = 30 + 50e^{kt}$$

$$40 = 30 + 50e^{\ln(0.8)t}$$

$$e^{\ln(0.8)t} = \frac{10}{50}$$

$$\ln\left(e^{\ln(0.8)t}\right) = \ln 0.2$$

$$\ln(0.8)t = \ln 0.2$$

$$t = \frac{\ln 0.2}{\ln 0.8}$$

$$t \simeq 7.2\text{mins.}$$

Example (2)

A bottle of soda at room temperature $72°$F is placed in a refrigerator where the temperature is $44°$F. After half an hour the soda has cooled to $61°$F.

(a) What is the temperature of the soda after another half hour?

(b) How long does it take for the soda to cool to $50°$F?

Solution We have the Newtons Law of Cooling is given from

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_s) \;\Rightarrow\; T = T_s + ce^{kt}.$$

Also we have the initial condition $T(0) = 72°$, the temperature of the surrounding $T_s = 44°$, and the additional condition $T(30) = 61°$

$$T = T_s + ce^{kt} \;\Rightarrow\; T(0) = 44 + ce^0 \;\Rightarrow\; 72 = 44 + c \;\Rightarrow\; c = 28.$$

Thus,

$$T = 44 + 28e^{kt}.$$

Now we will find the additional constant k by using the additional condition $T(1) = 70°$

$$T = 44 + 50e^{kt},$$

$$T(30) = 44 + 28e^{30k} \;\Rightarrow\; 61 = 44 + 28e^{30k} \;\Rightarrow\; e^{30k} = \frac{17}{28}$$

$$\ln\left(e^{30k}\right) = \ln 0.607$$

$$30k = \ln(0.607) \;\Rightarrow\; k = -0.0166$$

Thus, the temperature of the water at any time is given by

$$T(t) = 44 + 28e^{-0.0166t}.$$

(a) When $t = 60$ we have the temperature of the water

$$T(60) = 44 + 28e^{60\times(-0.0166)}$$

$$T(60) \simeq 54.3$$

(b) We have $T(t) = 50$ when

$$50 = 44 + 28e^{-0.0166t},$$

$$e^{-0.0166t} = \frac{6}{28}$$

$$\ln\left(e^{-0.0166t}\right) = \ln$$

$$-0.0166t = \ln 0.214$$

$$t = \frac{\ln 0.214}{-0.0166}$$

$$t \simeq 92.9 \text{ mins.}$$

Thus, the soda cools to $50°$F after about 1 hour 33 minutes.

Exercise

A small metal bar, whose initial temperature was $20°C$, is dropped in to large container of boiling water. How long will it take the bar to reach $90°C$ if it is known that its temperature increase $2°C$ per second? How will it take the bar to reach $98°C$.

Linear Differential Equations of Higher Order

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Linear Differential Equations of Higher Order

- General Solution of homogeneous linear differential equations
  - Initial-Value Problem (IVP)
  - Boundary-Value Problem (BVP)
  - Existence and Uniqueness of the Solution to an IVP
  - Linear Dependence and Independence of Functions
  - Criterion of Linearly Independent Solutions
  - Fundamental Set of Solutions
- Reduction of order Method (when one solution is given)
- Homogeneous Linear Differential Equations with Constant Coefficients
- Cauchy-Euler Differential Equation
- General Solutions of Nonhomogeneous Linear Differential Equations
  - Undetermined coefficients
  - Variation of Parameters

General Solution of Homogeneous Linear Differential Equations

Definition

The general linear differential equations of order n is an equation that can be written

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \qquad (1)$$

where $a_n(x),\ a_{n-1}(x),\ a_1(x)$ and $a_0(x)$ are functions of $x \in \mathrm{I} = (a, b)$, and they are called **coefficients**.
Equation (1) is called homogeneous linear differential equation if the function $g(x)$ is zero for all $x \in (a, b)$.
If $g(x)$ is not equal to zero on I, the equation (1) is called non-homogeneous linear differential equation.

Initial-Value Problem (IVP)

An n-th order initial-value problem associate with (1) takes the form:
Solve:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x),$$

subject to:

$$y(x_0) = y_0,\ y'(x_0) = y_1,\ y''(x_0) = y_2,\ \ \ldots,\ y^{n-1}(x_0) = y_{n-1}. \qquad (2)$$

Here (2) is a set of **initial conditions**.

Boundary-Value Problem (BVP)

Remark (Initial vs. Boundary Conditions)

Initial Conditions: all conditions are at the **same** $x = x_0$.
Boundary Conditions: conditions can be at **different** x.

Remark (Number of Initial/Boundary Conditions)

Usually a n-th order ODE requires n initial/boundary conditions to specify an unique solution.

Remark (Order of the derivatives in the conditions)

Initial/boundary conditions can be the value or the function of 0-th to $(n-1)$-th order derivatives, where n is the order of the ODE.

Example (Second-Order ODE)

Consider the following second-order ODE

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{d^{'}y}{dx^{'}} + a_0(x)y = g(x), \tag{3}$$

- IVP: Solve (3) s.t. $y(x_0) = y_0; y^{'}(x_0) = y_1$.
- BVP: Solve (3) s.t. $y(a) = y_0; y(b) = y_1$.
- BVP: Solve (3) s.t. $y(b) = y_0; y(a) = y_1$.

Existence and Uniqueness of the Solution to an IVP

Theorem

For the given linear differential equations of order n

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \qquad (4)$$

which is normal on an interval I*. Subject to*

$$y(x_0) = y_0,\ y'(x_0) = y_1,\ y''(x_0) = y_2,\ \ldots,\ y^{n-1}(x_0) = y_{n-1}. \qquad (5)$$

If $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ *and* $R(x)$ *are all continuous on an interval* I*,* $a_n(x)$ *is not a zero function on* I*, and the initial point* $x_0 \in I$*, then the above IVP has a unique solution in* I*.*

Example (1)

Discuss the Existence of unique solution of IVP

$$\begin{cases} (x^2+1)y'' + x^2y' + 5y = \cos(x) \\ y(3) = 2 \quad , \quad y'(3) = 1. \end{cases}$$

Solution The functions

$$a_2(x) = x^2 + 1, a_1(x) = x^2, a_0(x) = 5.$$

and

$$R(x) = \cos(x).$$

are continuous on $I = \mathbb{R} = (-\infty, \infty)$ and $a_2(x) \neq 0$ for all $x \in \mathbb{R}$, the point $x_0 = 3 \in I$. Then the previous Theorem assures that the IVP has a unique solution on $\mathbb{R}$.

Example (2)

Find an interval I for which the initial values problem (IVP)

$$\begin{cases} x^2y'' + \frac{x}{\sqrt{2-x}}y' + \frac{2}{\sqrt{x}}y = 0 \\ y(1) = 0 \quad , \quad y'(1) = 1. \end{cases} .$$

has a unique solution around $x_0 = 1$.

Solution The function

$$a_2(x) = x^2,$$

is continuous on $\mathbb{R}$ and $a_2(x) \neq 0$ if $x > 0$ or $x < 0$. But $x_0 = 1 \in I_1 = (0, \infty)$. The function

$$a_1(x) = \frac{x}{\sqrt{2-x}} ,$$

is continuous on $I_2 = (-\infty \ , 2)$ and the function

$$a_0(x) = \frac{2}{\sqrt{x}},$$

is continuous on $I_1 = (0, \infty)$.

Then the $(IVP\ )$has a unique solution on $I_1 \cap I_2 = (0, 2) = I$. We can take any interval $I_3 \subset (0, 2)$ such that $x_0 = 1 \in I_3$. So I is that the largest interval for which the (IVP) has a unique solution.

Example (3)

Find an interval I for which the IVP

$$\begin{cases} (x-1)(x-3)y'' + xy' + y = x^2 \\ y(2) = 1 \quad , \quad y'(2) = 0 \end{cases} .$$

has a unique solution about $x_0 = 2$.

Solution The functions

$$a_2(x) = (x-1)(x-3)\ ,\ a_1(x) = x\ , a_0(x) = 1\ , R(x) = x^2,$$

are continuous on $\mathbb{R}$. But $a_2(x) \neq 0$ if $x \in (-\infty, 1)$ or $x \in (1,3)$ or $x \in (3, \infty)$. As $x_0 = 2$ so we take $I = (1,3)$. Then the IVP has a unique solution on $I = (1,3)$

Exercises

1. Discuss the Existence of unique solution of IVP

$$\begin{cases} (|x|+3)y''+x^3y'+5y=\sin(x) \\ y(2)=1 \quad , \quad y'(2)=0. \end{cases}$$

2. Find an interval I for which the IVP

$$\begin{cases} (x-2)y''+3y=x \\ y(0)=0 \quad , \quad y'(0)=1 \end{cases}.$$

has a unique solution about $x_0=0$.

3. Find an interval I for which the IVP

$$\begin{cases} y''+(\tan x)y=e^x \\ y(0)=1 \quad , \quad y'(0)=0 \end{cases}.$$

has a unique solution about $x_0=0$.

Linear Dependence and Independence of Functions

Definition

A set of functions $\{f_1(x), f_2(x) \ldots, f_n(x)\}$ are **linearly dependent** on an interval I if $\exists\ c_1, c_2, \ldots, c_n$ not all zero i.e. $(c_1, c_2, \ldots, c_n) \neq (0, 0, \ldots, 0)$ such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0, \quad \forall\ x \in I$$

that is, the linear combination is a zero function.

If the set of functions is not linearly dependent, it is **linearly independent**, i.e. when $c_1, c_2, \ldots, c_n$ all zero i.e. $(c_1, c_2, \ldots, c_n) = (0, 0, \ldots, 0)$.

Example (1)

Show that $f_1(x) = \cos(2x)$, $f_2(x) = 1$, $f_3(x) = \cos^2(x)$ are linearly dependent on $\mathbb{R}$.

Solution We know that

$$f_3(x) = \cos^2(x) = \frac{1 + \cos(2x)}{2} = \frac{1}{2}\, f_2(x) + \frac{1}{2}\, f_1(x)$$

for all $x \in \mathbb{R}$. Then there exist $c_1 = c_2 = \frac{1}{2}$ and $c_3 = -1$ such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \qquad \text{for all } x \in \mathbb{R}.$$

So f_1 , f_2 and f_3 are linearly dependent on $\mathbb{R}$.

Example (2)

Show that

$$f_1(x) = 1,\ \ f_2(x) = \sec^2(x) \text{ and } \ \ f_3(x) = \tan^2(x)$$

are linearly dependent on $\left(0, \frac{\pi}{2}\right)$.

Solution We know that

$$f_2(x) = \sec^2(x) = 1 + \tan^2(x) = f_1(x) + \ \ f_3(x)$$

hence

$$f_1(x) - \ \ f_2(x) + \ \ f_3(x) = 0 \ \ \text{for all } x \in \left(0, \frac{\pi}{2}\right).$$

So there exist $c_1 = c_3 = 1$ and $c_2 = -1$ such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \qquad \text{for all } \ x \in \left(0, \frac{\pi}{2}\right).$$

So f_1 , f_2 and f_3 are linearly dependent on $\left(0, \frac{\pi}{2}\right)$.

Example (3)

Show that $f_1(x) = x$ and $f_2(x) = x^2$ are linearly independent on $I = [-1, 1]$.

Solution Let c_1 , $c_2 \in \mathbb{R}$ such that

$$c_1 f_1(x) + c_2 f_2(x) = 0, \quad \text{for all } x \in I \ .$$

We have to prove that $c_1 = c_2 = 0$. As

$$c_1 x + c_2 x^2 = 0 \quad \text{for all } -1 \leq x \leq 1,$$

then for $x = 1$ and $x = -\frac{1}{2}$ we have

$$c_1 + c_2 = 0,$$

and

$$-\frac{1}{2} c_1 + \frac{1}{4} c_2 = 0,$$

which implies that $c_1 = c_2 = 0$. Then f_1 and f_2 are linearly independent on I.

Example (4)

Show that

$$f_1(x) = \sin(x)\ , f_2(x) = \sin(2x).$$

are linearly independent on $I = [0, \pi)$.

Solution Let c_1 , $c_2 \in I$ such that

$$c_1 f_1(x) + c_2 f_2(x) = 0 \quad \text{for all} x \in I.$$

We have to show that $c_1 = c_2 = 0$. In fact for $x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$ we have

$$\begin{cases} c_1 \sin\left(\frac{\pi}{4}\right) + c_2 \sin\left(\frac{\pi}{2}\right) = 0 \\ c_1 \sin\left(\frac{\pi}{3}\right) + c_2 \sin\left(2\frac{\pi}{3}\right) = 0 \end{cases}$$

hence

$$\frac{1}{\sqrt{2}} c_1 + c_2 = 0 \quad , \quad \frac{\sqrt{3}}{2} c_1 + \frac{\sqrt{3}}{2} c_2 = 0,$$

which implies that $c_1 = c_2 = 0$. Then f_1 and f_2 are linearly independent on I.

Example (5)

Show that

$$f_1(x) = x^2 \text{ and } f_2(x) = x\,|x|$$

(i) linearly dependent on $[0, 1]$
(ii) linearly independent on $[-1, 1]$

Solution

(i) on $[0, 1]$ we have

$$f_1(x) = f_2(x) = x^2,$$

hence

$$f_1(x) - f_2(x) = 0 \quad \text{for all} \quad 0 \leq x \leq 1.$$

So there exist $c_1 = 1$, $c_2 = -1$ such that

$$c_1 f_1(x) + c_2 f_2(x) = 0 \qquad \text{for all} \quad 0 \leq x \leq 1.$$

Then f_1 and f_2 are linearly dependent on $[0, 1]$.

(ii) Let c_1 , $c_2 \in \mathbb{R}$ such that

$$c_1 f_1(x) + c_2 f_2(x) = 0 \quad \text{for all} \quad -1 \leq x \leq 1,$$

hence

$$c_1 x^2 + c_2 x\,|x| = 0 \text{ for all } -1 \leq x \leq 1.$$

Now for $x = 1$ and $x = -1$ we have $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$ which implies that $c_1 = c_2 = 0$. Then f_1 and f_2 are linearly independent on $[-1, 1]$.

Exercises

1. Determine whether the functions

$$f_1(x) = x \ , f_2(x) = x^2, \quad f_3(x) = 4x - x^2.$$

are linearly dependent or independent on $(-\infty, \infty)$.

2. Determine whether the functions

$$f_1(x) = e^x \ , f_2(x) = e^{-x}, \quad f_3(x) = \cosh x.$$

are linearly dependent or independent on $(-\infty, \infty)$.

3. Determine whether the functions

$$f_1(x) = x \ , f_2(x) = x^2 - 1, \quad f_3(x) = x^2 + 2x + 1.$$

are linearly dependent or independent on $[0, 1]$.

Criterion of Linearly Independent Solutions

Consider the homogeneous linear n-th order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0,$$

Given n solutions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$, we would like to test if they are independent or not.

Note: In Linear Algebra, to test if n vectors $\{v_1, v_2, \ldots, v_n\}$ are linearly independent, we can compute the determinant of the matrix.

$$\mathrm{V} := [v_1 \ v_2 \ \ldots \ v_n].$$

If the determinant of $\mathrm{V} = 0$, they are linearly dependent; if the determinant of $\mathrm{V} \neq 0$, they are linearly independent.

Definition

For n functions $W(f_1, f_2, \ldots, f_n)$ which are $n-1$ times differentiable on an interval I, the **Wronskian** $W(x, f_1, f_2, \ldots, f_n)$ as a function on I is defined by

$$W(x, f_1, f_2\ , ..., f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ f_1'' & f_2'' & \cdots & f_n'' \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{n-1} & f_2^{n-1} & .... & f_n^{n-1} \end{vmatrix}$$

To test the linear independence of n solutions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ to

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0, \qquad (6)$$

we can use the following theorem.

Theorem

Let $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ be n solutions to the homogeneous linear DE (6) on an interval I. They are **linearly independent** *on I*

$$\iff W(x, f_1, f_2\ , ..., f_n) := \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ f_1'' & f_2'' & \cdots & f_n'' \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{n-1} & f_2^{n-1} & .... & f_n^{n-1} \end{vmatrix} \neq 0.$$

Example (1)

Prove that $f_1(x) = x^2$, $f_2(x) = x^2 \ln(x)$ are linearly independent on $(0, \infty)$.

Solution We have that

$$\begin{aligned} W(f_1, f_2) &= \begin{vmatrix} x^2 & x^2 \ln(x) \\ 2x & 2x \ln(x) + x \end{vmatrix} \\ &= 2x^3 \ln(x) + x^3 - 2x^3 \ln(x) = x^3 \neq 0 \end{aligned}$$

for all $x \in (0, \infty)$,

then f_1 and f_2 are linearly independent on $(0, \infty)$.

Example (2)

It is easy to see that the functions

$$y_1 = x \ , y_2 = x^2,$$

and

$$y_3 = x^3.$$

are solutions of the differential equation

$$x^3y''' - 3x^2y'' + 6xy' - 6y = 0.$$

Show that y_1 , y_2 and y_3 are linearly independent on $(0, \infty)$.

Solution Here we have $a_3(x) = x^3 \neq 0$ for all $x > 0$ or $x < 0$. By using the Wronskian we have

$$W(y_1, y_2 \ , \ y_3) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3 \neq 0.$$

for all $x \in (0, \infty)$ or for all $x \in (-\infty, 0)$. So y_1 , y_2 and y_3 are linearly independent on $(0, \infty)$.

Exercises

1. Show by computing the Wronskian that the functions

$$f_1(x) = x \ , f_2(x) = xe^x, \quad f_3(x) = x^2e^x.$$

are linearly dependent or independent on $(0, \infty)$.

2. Show that the functions

$$y_1 = \cosh(2x)$$

and

$$y_2 = \sinh(2x)$$

are solutions of the differential equation

$$y'' - 4y' = 0.$$

Show that y_1 and y_2 are linearly independent on $(-\infty, \infty)$.

Fundamental Set of Solutions

Definition

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0, \qquad (7)$$

Any set $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ of n linearly independent solutions to the homogeneous linear n-th order DE (7) on an interval I is called a **fundamental set of solutions**.

Theorem

Let $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ be a fundamental set of solutions to the homogeneous linear n-th order DE (7) on an interval I. Then the **general solution to** *(7) is*

$$y(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x),$$

where $\{c_i \mid (i = 1, 2, \ldots, n)\}$ are arbitrary constants.

Example (1)

Verify that $y_1 = e^{2x}$ and $y_2 = e^{-3x}$ form a fundamental set of solutions of the differential equation

$$y'' + y' - 6y = 0.$$

and find the general solution.

Solution Substituting

$$y_1 = e^{2x}, y_1' = 2e^{2x}, y_1'' = 4e^{2x},$$

in the differential equation we have

$$4e^{2x} + 2e^{2x} - 6e^{2x} = 0.$$

Hence $y_1 = e^{2x}$ is a solution of the differential equation. By the same method we can prove that $y_2 = e^{-3x}$ is also a solution of the differential equation.

Now we have

$$W(e^{2x}, e^{-3x}) = \begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & -3e^{-3x} \end{vmatrix} = -5e^{-x} \neq 0 \text{ for all } x \in \mathbb{R}.$$

Then y_1 and y_2 are linearly independent on $\mathbb{R}$. From the previous Theorem we deduce the general solution of the differential equation given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

where c_1 ,$c_2 \in \mathbb{R}$.

Example (2)

It is easy to see that the functions

$$y_1 = e^x, y_2 = e^{2x}, \quad \text{and } y_3 = e^{3x}$$

are solutions of the differential equation

$$y''' - 6y'' + 11y' - 6y = 0.$$

Find the general solution of the differential equation.

Solution Since

$$W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0.$$

for all $x \in \mathbb{R}$.

We deduce that

$$y(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$

is the general solution of the differential equation.

Example (3)

Prove that

$$y_1 = x^3 e^x, \text{ and } y_2 = e^x.$$

are solutions of the differential equation

$$xy'' - 2(x+1)y' + (x+2)y = 0$$

where $x > 0$. Find also the general solution of the differential equation.

Solution Substituting

$$y_1 = x^3 e^x, \; y_1' = 3x^2 e^x + x^3 e^x, \; y_1'' = 6xe^x + 6x^2 e^x + x^3 e^x,$$

in the differential equation we have

$$6x^2 e^x + 6x^3 e^x + x^4 e^x - 6x^3 e^x - 2x^4 e^{xe^x} - 6x^2 e^x + -2x^3 e^x + x^4 e^x + 2x^3 e^x = 0.$$

Substituting

$$y_2 = y_2' = y_2'' = e^x,$$

in the differential equation

$$xe^x - 2xe^x - 2e^x + xe^x + 2e^x = 0.$$

Now we have to show that

$$y_1 = x^3e^x,$$

and

$$y_2 = e^x.$$

are linearly independent on $(0\ ,\ \infty)$.

In fact

$$W(x^3e^x, e^x) = \begin{vmatrix} x^3e^x & e^x \\ 3x^2e^x + x^3e^x & e^x \end{vmatrix} = -3x^2e^x \neq 0 \quad \text{for all} \quad x > 0.$$

Then

$$y_1 = x^3e^x,$$

and

$$y_2 = e^x.$$

are linearly independent on $(0\ ,\ \infty)$ and we conclude that

$$y_c = c_1 x^3 e^x + c_2 e^x.$$

is the general solution of the differential equation.

Exercises

Verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval, then find the general solution of the differential equation.

- $y'' - y' - 12y = 0; \quad e^{-3x}, \ e^{4x}$ on $(-\infty, \infty)$
- $x^3y''' + 6x^2y'' + 4xy' - 4y = 0; \quad x, \ x^{-2}, \ x^{-2}\ln x$ on $(0, \infty)$
- $y^{(4)} + y'' = 0; \quad 1, \ x, \ \sin x, \ \cos x$ on $(0, \infty)$

Reduction of order Method (when one solution is given)

It is employed when one solution $y_1(x)$ is known and a second linearly independent solution $y_2(x)$ is desired. The method also applies to n-th order equations.

Suppose that $y_1(x)$ is a non-zero solution of the equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \tag{8}$$

where $a_0(x)$, $a_1(x)$ and $a_2(x)$ are continuous functions defined on interval I sauch that $a_2(x) \neq 0$ for all $x \in I$.

The method of reduction of order is used to obtain a second linearly independent $y_2(x)$ solution to this differential equation (8) using our one known solution.

We suppose that the solution of (8) is in the form

$$y = u(x)y_1,$$

where u is a fountain of x and which will be determined and satisfies a linear second-order differential equation (8) by using the following method

$$y = u(x)y_1 \Rightarrow y' = u'y_1 + y_1'u \Rightarrow y'' = u''y_1 + 2u'y_1' + y_1''u.$$

It is best to describe the procedure with a concrete example.

Example (1)

If

$$y_1 = \frac{\sin x}{\sqrt{x}}.$$

is a solution of the differential equation

$$4x^2y'' + 4xy' + (4x^2 - 1)y = 0 \quad on\ 0 < x < \pi.$$

then find the general solution of the differential equation..

Solution The solution of the differential equation is of the form $y = u(x)y_1$ or

$$y = \frac{\sin x}{\sqrt{x}}u \ = (\sin x)\,(x)^{\frac{-1}{2}}\,u,$$

hence

$$y' = (\cos x)(x)^{\frac{-1}{2}}u - \frac{1}{2}\sin x(x)^{\frac{-3}{2}}u + \sin x(x)^{\frac{-1}{2}}u',$$

$$\begin{aligned} y'' &= -\sin x(x)^{\frac{-1}{2}}u - \cos x(x)^{\frac{-3}{2}}u + 2\cos x(x)^{\frac{-1}{2}}u' \\ &\quad + \frac{3}{4}\sin x(x)^{\frac{-5}{2}}u - \sin x(x)^{\frac{-3}{2}}u' + \sin x(x)^{\frac{-1}{2}}u'' \end{aligned}$$

we substitute y , y', and y'' in the arbitrary constant we obtain

$$4x^{\frac{3}{2}}\sin xu'' + \left(8x^{\frac{3}{2}}\cos x\right)u' = 0,$$

hence

$$\sin xu'' + 2\cos xu' = 0.$$

To solve this differential equation we put $w = u'$, then we have $w' = u''$.

Then

$$\int \frac{dw}{w}dx + \int \frac{2\cos x}{\sin x}dx = 0,$$

hence

$$u' = w = \frac{c_1}{\sin^2 x},$$

where $c_1 \neq 0$ is an arbitrary constant. So we have $u = -c_1 \cot x + c_2$, hence

$$y = y_1 u = \frac{\sin x}{\sqrt{x}}(-c_1 \cot x + c_2),$$

or

$$y = c_3 \frac{\cos x}{\sqrt{x}} + c_2 \frac{\sin x}{\sqrt{x}},$$

finally we have

$$y = c_2 y_1 + c_3 y_2,$$

where $c_3 = -c_1$ and c_2 are arbitrary constants, is the general solution of the differential equation and we can prove that

$$y_1 = \frac{\sin x}{\sqrt{x}} \text{ and } y_2 = \frac{\cos x}{\sqrt{x}}$$

are linearly independent on solutions $(0, \pi)$.

General case of Equation (8)

Equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,$$

can be written as the form

$$y'' + p(x)y' + q(x)y = 0, \tag{9}$$

where

$$p(x) = \frac{a_1(x)}{a_2(x)},$$

and

$$q(x) = \frac{a_0(x)}{a_2(x)}.$$

Also, let us suppose that y_1 is a known solution of (9) on I and $y_1(x) \neq 0$ for all $x \in I$.

Thus the second solution of (9) y_2 can be given from

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2}\, dx. \tag{10}$$

Example (1)

If

$$y_1 = \frac{\sin x}{\sqrt{x}}$$

is a solution of the differential equation

$$4x^2y'' + 4xy' + (4x^2 - 1)y = 0 \quad \text{on} \quad (0, \pi),$$

then find the second solution .

Solution As

$$y'' + \frac{1}{x}y' + \frac{4x^2 - 1}{4x^2}y = 0.$$

then

$$p(x) = \frac{1}{x},$$

and

$$e^{-\int p(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}.$$

We have

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx = \frac{\sin x}{\sqrt{x}} \int \frac{\frac{1}{x}}{\frac{\sin^2(x)}{x}} dx, \\ &= \frac{\sin x}{\sqrt{x}} \int \frac{dx}{\sin^2(x)} = \frac{-\cos x}{\sqrt{x}}. \end{aligned}$$

Hence

$$y_2 = \frac{-\cos x}{\sqrt{x}} \quad \text{or} \quad y_2 = \frac{\cos x}{\sqrt{x}}$$

is the second solution of the differential equation on $(0, \pi)$.

Example (2)

If $y_1 = e^{3x}$ is a solution of the differential equation

$$xy'' + (x-1)y' + (3-12x)y = 0 \quad ; \quad x > 0.$$

Find the general solution.

Solution We have

$$y'' + (1 - \frac{1}{x})y' + (\frac{3}{x} - 12)y = 0.$$

From the formula (10) we can find directly y_2, where

$$\int -p(x)dx = \int (-1 + \frac{1}{x})dx = -x + \ln x,$$

hence

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx = e^{3x} \int \frac{e^{-x+\ln x}}{e^{6x}} dx = e^{3x} \int x e^{-7x} dx \\ &= e^{3x} \left[-\frac{1}{7} x e^{-7x} - \frac{1}{49} e^{-7x} \right] \\ &= e^{-4x} \left(\frac{-x}{7} - \frac{1}{49} \right) = -\frac{1}{7} e^{-4x} \left(x + \frac{1}{7} \right). \end{aligned}$$

Then the general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{3x} + c_2 e^{-4x} \left(x + \frac{1}{7} \right)$$

on the interval $(0, \infty)$.

Exercises

Apply the reduction of order method to obtain another linearly independent solution for the following differential equations

- $y'' - 4y' + 4y = 0; \quad y_1 = e^{2x}.$
- $x^2y'' + x^2y' - (x+2)y = 0; \quad y_1 = x^{-1}e^{-x}, \quad x > 0.$
- $x^2(1 - \ln x)y'' + xy' - y = 0; \quad y_1 = x, \quad x > e.$

Homogeneous Linear Differential Equations with Constant Coefficients

The linear differential equations with Constant Coefficients has the general form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0, \tag{11}$$

which is a homogeneous linear DE with **constant real coefficients**, where each coefficient $a_i, 1 \leq i \leq n$ is real constant and $a_n \neq 0$.

Definition

The polynomial

$$f(m) = a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0, \tag{12}$$

is called the characteristic polynomial for equation (11), and $f(m) = 0$ is called the characteristic equation of the linear differential equations with constant coefficients (11).

We conclude that if m is a root of equation (52), then

$$y = e^{mx}$$

is a solution of the differential equation (11). Also, Equation (52) has n roots.

Let us summarize the method to solve the differential equation (11)

(1) If all the roots of the characteristic equation are **real roots** then:
(i) If the roots are distinct (i.e. $m_1 \neq m_2 \neq m_3 \neq \cdots \neq m_n$), then the solution of the differential equation (11) is given by

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

(ii) If the roots are equal (i.e. $m_1 = m_2 = m_3 = \cdots = m_n$) (i.e. $m = m_i$ is a root of multiplicity n), then the solution of the differential equation (11) is given by

$$y = c_1 e^{mx} + c_2 x e^{mx} + c_3 x^2 e^{mx} + \cdots + c_n x^{n-1} e^{mx}$$

$$y = (c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1}) e^{mx}$$

Example (1)

Solve the differential equation

$$y'' - y = 0.$$

Solution For this, the characteristic equation is $m^2 - 1 = 0$ hence $m = \mp 1$.Then $y_1 = e^x$ and $y_2 = e^{-x}$ form the fundamental set of solutions, hence the general solution is

$$y = c_1 e^x + c_2 e^{-x}.$$

Example (2)

Find the general solution of the differential equation

$$y''' - 6y'' + 11y' - 6y = 0 \ .$$

Solution For this differential equation the characteristic equation is

$$m^3 - 6m^2 + 11m - 6 = (m-1)(m-2)(m-3) = 0.$$

Then $m = 1, 2, 3$ and $y_1 = e^x$, $y_2 = e^{2x}$ and $y_3 = e^{3x}$ form the fundamental set of solutions, hence the general solution is

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$

Example (3)

Solve the differential equation

$$y'' - 2y' + y = 0.$$

Solution The characteristic equation for the differential equation is

$$m^2 - 2m + 1 = 0 \ ,$$

so $m = 1$ is a root of multiplicity 2, hence the general solution is

$$y = c_1 e^x + c_2 x e^x.$$

Example (4)

Solve the differential equation

$$y''' - 3y'' + 3y' - y = 0$$

Solution The characteristic equation for the differential equation is $m^3 - 3m^2 + 3m - 1 = (m-1)^3 = 0$, so $m = 1$ is a root of multiplicity 3 then the general solution is

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x.$$

Now we see the second case

(2) If the characteristic equation has **complex conjugate roots** such as

$$m = \alpha \mp i\beta$$

then he solution of the differential equation of second order is given by

$$y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

Remember:

$$1)\ \sqrt{-1} = i$$

$$2)\ x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

to find the roots of Quadratic equation

$$ax^2 + bx + c = 0$$

Example (5)

Solve the differential equation

$$y'' + 4y' + 5y = 0.$$

Solution The characteristic (auxiliary) equation for the differential equation is $m^2 + 4m + 5 = 0$, now we need to find the roots of this characteristic equation

$$m = \frac{-4 \mp \sqrt{16 - 20}}{2}$$

then $m = -2 \mp i$ hence the general solution is

$$y(x) = c_1 e^{-2x} \cos(x) + c_2 e^{-2x} \sin(x).$$

Example (6)

Solve the differential equation

$$y^{(5)} - 3y^{(4)} + 4y''' - 4y'' + 3y' - y = 0.$$

Solution The characteristic for the differential equation is

$$m^5 - 3m^4 + 4m^3 - 4m^2 + 3m - 1 = 0,$$

then

$$m^5 - 3m^4 + 4m^3 - 4m^2 + 3m - 1 = (m-1)^3(m^2+1) = 0.$$

Thus $m = 1 \ , 1 \ , 1 \ , \mp i$ where $\sqrt{-1} = i$ and the general solution of the equation has the form

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + c_4 \cos x + c_5 \sin x.$$

Example (7)

Solve the initial value problem (IVP)

$$\begin{cases} y'' + y' + y = 0 \\ y(0) = 1 \quad , \quad y'(0) = \sqrt{3}. \end{cases}$$

Solution The characteristic equation for the differential equation is

$$m^2 + m + 1 = 0.$$

Hence

$$m = -\frac{1}{2} \mp \frac{\sqrt{3}}{2} i.$$

So the general solution of a differential equation is

$$y = c_1 e^{\frac{-x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{\frac{-x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right).$$

from the conditions $y(0) = 1$ and $y'(0) = \sqrt{3}$ we have $c_1 = 1$. and

$$\frac{-c_1}{2} + c_2\frac{\sqrt{3}}{2} = \sqrt{3}$$

hence $c_1 = 1$ and $c_2 = 2 + \frac{1}{\sqrt{3}}$. So the solution of the IVP is

$$y = e^{\frac{-x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right) + (2 + \frac{1}{\sqrt{3}})e^{\frac{-x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right).$$

Exercises

Find the general solution of the following differential equations

- $2y'' + 3y' + y = 0$
- $y'' - y' - 6y = 0$
- $y''' - 4y'' - 5y' = 0$
- $y^{(4)} - 2y'' + y = 0$
- $2y^{(5)} - 7y^{(4)} + 12y''' + 8y'' = 0$

Find the solution of the initial value problems

- $y'' + y' + 2y = 0; \quad y(0) = y'(0) = 0$
- $y''' + 12y'' + 36y' = 0; \quad y(0) = 0,\ y'(0) = 1,\ y''(0) = -7$

Cauchy-Euler Differential Equation

A Cauchy-Euler differential equation is in the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = 0, \qquad (13)$$

where each coefficient $a_i, 1 \leq i \leq n$ are constants and $a_n \neq 0$ i.e. the coefficient $a_n x^n$ should never be zero. Equation (13) is on the interval either $(0, \infty)$ or $(-\infty, 0)$.

Euler differential equation is probably the simplest type of linear differential equation with variable coefficients.

The most common Cauchy-Euler equation is the second-order equation, appearing in a number of physics and engineering applications, such as when solving Laplace's equation in polar coordinates.

It is given by the equation

$$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0 \tag{14}$$

To solve the Cauchy-Euler differential equation, we assume that $y = x^m$, where $x > 0$ and m is a root of a polynomial equation.

Example (1)

Solve the Cauchy-Euler differential equation

$$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0.$$

Solution We substitute

$$y = x^m \implies y' = mx^{m-1} \implies y'' = m(m-1)x^{m-2}$$

in the differential equation, we obtain

$$x^2[m(m-1)x^{m-2}] + ax[mx^{m-1}] + bx^m = 0$$

$$x^m(m^2 - m) + amx^m + bx^m = 0$$

$$x^m[(m^2 - m) + am + b] = 0$$

$$x^m[m^2 + (1-a)m + b] = 0.$$

Since $x^m \neq 0$, then we have

$$m^2 + (1-a)m + b = 0$$

We then can solve for m. There are three particular cases of interest:

Case 1: Two distinct roots, m_1 and m_2. Thus, the solution is given by

$$y = c_1 x^{m_1} + c_2 x^{m_2}.$$

Case 2: One real repeated root, m. Thus, the solution is given by

$$y = c_1 x^m \ln(x) + c_2 x^m.$$

Case 3: Complex roots, $\alpha \pm i\beta$. Thus, the solution is given by

$$y = c_1 x^\alpha \cos(\beta \ln(x)) + c_2 x^\alpha \sin(\beta \ln(x)).$$

Example (2)

Solve the Euler differential equation

$$2x^2y'' - 3xy' - 3y = 0. \tag{15}$$

For $x > 0$.

Solution We substitute

$$y = x^m \implies y' = mx^{m-1} \implies y'' = m(m-1)x^{m-2}$$

in the differential equation, we obtain

$$2x^2[m(m-1)x^{m-2}] - 3x[mx^{m-1}] - x^m = 0$$

$$x^m(2m^2 - 2m) - 3mx^m - 3x^m = 0$$

$$x^m[2m^2 - 2m - 3m - 3] = 0$$

$$x^m[2m^2 - 5m - 3] = 0.$$

Since $x^m \neq 0$, then we have

$$2m^2 - 5m - 3 = 0$$

So the roots of this equation are $m_1 = -\frac{1}{2}$, $m_2 = 3$ .Thus, from case 1 we have the solution is given by

$$y(x) = c_1 x^{-\frac{1}{2}} + c_2 x^3.$$

which is the general solution.

Example (3)

Find the general of the differential equation

$$x^2y'' - 3xy' + 13y = 0 \quad ; \quad x > 0.$$

Solution Substituting $y = x^m$ in the equation, we obtain

$$m(m-1) - 3m + 13 = m^2 - 4m + 13 = 0.$$

Then we have two complex roots $m = 3 \mp 3i$ (case 3), hence the the general of the differential equationis

$$y = c_1x^3\cos(3\ln x) + c_2x^3\sin(3\ln x) \quad ; \quad x > 0.$$

If we suppose $x < 0$, then the general of the differential equation is

$$y = c_1(-x)^3\cos(3\ln(-x)) + c_2(-x)^3\sin(3\ln(-x)) \quad ; \quad x < 0.$$

Example (4)

Find the general solution of the differential equation

$$x^4y^{(4)} - 5x^3y''' + 3x^2y'' - 6xy' + 6y = 0 \quad ; \quad x > 0.$$

Solution Substituting $y = x^m$ in the equation, we obtain

$$m(m-1)(m-2)(m-3) - 5m(m-1)(m-2) + 3m(m-1) - 6m + 6 = 0.$$

This implies that

$$(m-1)(m-2)(m^2 - 8m + 3) = 0.$$

The roots of this equation are $m = 1$, $m = 2$, and $m = 4 \mp \sqrt{13}$, then the general solution of the differential equation is

$$y = c_1x + c_2x^2 + c_3x^{4+\sqrt{13}} + c_4x^{4-\sqrt{13}} \quad ; \quad x > 0.$$

Example (5)

Find the general solution of the differential equation

$$x^5 y^{(5)} - 2x^3 y''' + 4x^2 y'' = 0 \quad ; \quad x < 0.$$

Solution Substituting $y = x^m$ in the equation, we obtain

$$m(m-1)(m-2)(m-3)(m-4) - 2m(m-1)(m-2) + 4m(m-1) = 0,$$

$$m(m-1)(m^3 - 9m^2 + 24m - 20) = m(m-1)(m-2)^2(m-5) = 0.$$

So the roots of this equation are $m = 0$, $m = 1$, $m = 2$ repeated two times and $m = 5$, then the general of the differential equation is

$$y = c_1 + c_2(-x) + c_3(-x)^2 + c_4(-x)^2 \ln(-x) + c_5(-x)^5.$$

Exercises

Find the general solution of the following differential equations, where we suppose that $x > 0$.

- $x^2y'' - y = 0$
- $x^2y'' + 5xy' + 3y = 0$
- $4x^2y'' + 4xy' - y = 0$
- $x^3y''' + xy' - y = 0$
- $x^3y''' + 4x^2y'' - 8xy' + 8y = 0$
- $(3x+4)^2y'' + 10(3x+4)y' + 9y = 0;\ x > -\frac{4}{3}$

General Solutions of Nonhomogeneous Linear Differential Equations

Nonhomogeneous linear n-th order ODE takes the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \qquad (16)$$

where $a_n(x),\ a_{n-1}(x),\ a_1(x)$ and $a_0(x)$ are functions of $x \in \mathrm{I} = (a, b)$, such that $a_n(x) \neq 0$ for all $x \in I$, and $g(x) \neq 0$.

Idea

- Find the general solution y_c to the homogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

- Find a solution y_p to the nonhomogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

- The general solution $y = y_c + y_p$.

Undetermined coefficients

Let us take an example

Example (1)

Find the general solution of the differential equation :

$$y'' - y = -2x^2 + 5 + 2e^x. \qquad (*)$$

Solution

1) First we have to find the general solution of the differential equation :

$$y'' - y = 0.$$

For , we have $m^2 - 1 = 0$, hence $m = \mp 1$ then

$$y_c = c_1 e^x + c_2 e^{-x}.$$

2) The form of the particular solution of

$$y'' - y = -2x^2 + 5,$$

is

$$y_{1,p} = Ax^2 + Bx + C,$$

and the form of the particular solution of

$$y'' - y = 2e^x,$$

is

$$y_{2,p} = Dxe^x,$$

because $r = 1$ is a simple root of the characteristic equation. Thus the particular solution of (*) is

$$y_p = y_{1,p} + y_{2,p} = Ax^2 + Bx + C + Dxe^x.$$

Now we have to find the constants A, B, C , and D by substituting y_p and y_p'' in differential equation (*) and we find

$$y_p'' - y_p = -Ax^2 - Bx + 2A - C + 2De^x = -2x^2 + 5 + 2e^x.$$

Equating coefficients of similar terms (because the functions $x^2, 1$ and are e^x linearly independent on $\mathbb{R}$), we obtain the following system of equation $A = 2$, $B = 0$, $2A - C = 5$, and $2D = 2$. Thus we have $A = 2$, $B = 0$, $C = -1$, and $D = 1$.Then the particular solution of (*) is

$$y_p = 2x^2 - 1 + xe^x,$$

and the general solution of the differential equation of (*) is

$$y = y_c + y_p = c_1e^x + c_2e^{-x} + 2x^2 - 1 + xe^x.$$

Some of the Typical forms of the particular integral

Function of x	Form for y_p
ke^{ax}	Ce^{ax}
$kx^n,\ n = 0, 1, 2, \ldots$	$\sum_{i=1}^{n} C_i x^i$
$k\cos(ax)$ or $k\sin(ax)$	$C_1\cos(ax) + C_2\sin(ax)$
$ke^{ax}\cos(bx)$ or $ke^{ax}\sin(bx)$	$e^{ax}\left(C_1\cos(bx) + C_2\sin(bx)\right)$
$\left(\sum_{i=1}^{n} k_i x^i\right)\cos(ax)$ or $\left(\sum_{i=1}^{n} k_i x^i\right)\sin(ax)$	$\left(\sum_{i=1}^{n} C_i x^i\right)\cos(ax) + \left(\sum_{i=1}^{n} R_i x^i\right)\sin(ax)$

Exercises

Find the general solution of the following differential equations.

- $x^2y'' - y = 0$
- $y'' + 4y = sin(2x) + e^x$
- $y'' - 5y' + 4y = e^{2x}(\cos x + \sin x)$

Find only the form of the particular solution of the given differential equation by using the method of undetermined coefficients.

- $y'' - y = e^x + s \sin x$
- $y'' - y = x^2e^x$
- $y^{(6)} - 3y^{(3)} = 3x + 1$
- $y''' - y' = x^5 + \cos x$

Variation of Parameters

This method is used to determine the particular solution y_p of nonhomogeneous differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \qquad (17)$$

If we have the nonhomogeneous differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x), \qquad (18)$$

which has the particular solution

$$y_p = y_1 u_1 + y_2 u_2,$$

where y_1 and y_2 are the first and the second solution of the homogeneous differential equation, respectively.

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad (19)$$

Here we will explain the method to find u_1 and u_2. So, if we have y_1 & y_2 , then we will determine as below

$$W(x, y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1',$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ g(x) & y_2' \end{vmatrix} = -y_2 g(x),$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x) \end{vmatrix} = y_1 g(x).$$

Thus,

$$u_1' = \frac{W_1}{W}$$

and

$$u_2' = \frac{W_2}{W}.$$

Example (1)

Solve the differential equation

$$y'' + y = \csc x \quad ; \quad 0 < x < \pi.$$

Solution

1) The general solution of

$$y'' + y = 0,$$

is

$$y_c = c_1 \sin x + c_2 \cos x.$$

2) The particular solution of

$$y'' + y = \csc x,$$

is the form

$$y_p = y_1 u_1 + y_2 u_2,$$

where

$$y_1 = \sin x \quad and \quad y_2 = \cos x.$$

The functions u_1 and u_2 are determined from the system below

$$W(x, y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1,$$

$$W_1 = \begin{vmatrix} 0 & \cos x \\ \csc x & -\sin x \end{vmatrix} = -\cot x,$$

$$W_2 = \begin{vmatrix} \sin x & 0 \\ \cos x & \csc x \end{vmatrix} = 1,$$

Hence

$$u_1' = \frac{W_1}{W} = \cot x,$$

then

$$u_1 = \ln(\sin x).$$

But

$$u_2' = -1,$$

hence $u_2 = -x$. Therefore we have

$$y_p = y_1 u_1 + y_2 u_2 = \sin x . \ln(\sin x) - x \cos x,$$

and the general solution of the differential equation is

$$y = y_c + y_p = c_1 \sin x + c_2 \cos x + \sin x . \ln(\sin x) - x \cos x.$$

Example (2)

Solve the differential equation

$$y'' - 4y' + 4y = (x+1)e^{2x}.$$

Solution

1) The general solution of

$$y'' - 4y' + 4y = 0,$$

is

$$y_c = c_1 e^{2x} + c_2 x e^{2x}.$$

2) Let

$$y_1 = e^{2x} \quad and \quad y_2 = xe^{2x}.$$

So we have

$$W(x, y_1, y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x},$$

$$W_1(x, y_1, y_2) = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = -x(x+1)e^{4x},$$

and

$$W_2(x, y_1, y_2) = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x},$$

hence

$$u_1' = \frac{W_1}{W} = -x(x+1) = -x^2 - x,$$

so

$$u_1 = -\frac{x^3}{3} - \frac{x^2}{2}$$

But

$$u_2' = \frac{W_2}{W} = x + 1,$$

then

$$u_2 = \frac{x^2}{2} + x.$$

Therefore,

$$y_p = y_1u_1 + y_2u_2 = (-\frac{x^3}{3} - \frac{x^2}{2})e^{2x} + x(\frac{x^2}{2} + x)e^{2x} = (\frac{x^3}{6} + \frac{x^2}{2})e^{2x},$$

and The general solution of the differential equation is

$$y = y_c + y_p = c_1e^{2x} + c_2xe^{2x} + (\frac{x^3}{6} + \frac{x^2}{2})e^{2x}.$$

In this example we can use the undetermined coefficients, where

$$y_p = x^2(A + Bx)e^{2x}.$$

Example (3)

Solve the Differential equation

$$y'' - 3y' + 2y = \frac{1}{1 + e^{-x}}.$$

Solution

1) The general solution of

$$y'' - 3y' + 2y = 0.$$

is

$$y_c = c_1 e^x + c_2 e^{2x}.$$

2) Let

$$y_1 = e^x \quad \text{and } y_2 = e^{2x},$$

then

$$W(x, y_1, y_2) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x},$$

$$W_1(x, y_1, y_2) = \begin{vmatrix} 0 & e^{2x} \\ \frac{1}{1+e^{-x}} & 2e^{2x} \end{vmatrix} = \frac{-e^{2x}}{1+e^{-x}},$$

$$W_2(x, y_1, y_2) = \begin{vmatrix} e^{x} & 0 \\ e^{x} & \frac{1}{1+e^{-x}} \end{vmatrix} = \frac{e^{x}}{1+e^{-x}},$$

hence

$$u_1' = \frac{W_1}{W} = -\frac{e^{-x}}{1+e^{-x}}$$

and

$$u_1(x) = -\int \frac{e^{-x}}{1+e^{-x}}dx = \ln\left(1+e^{-x}\right).$$

But

$$u_2' = \frac{W_2}{W} = \frac{e^{-2x}}{1+e^{-x}},$$

and

$$u_2 = \int \frac{e^{-2x}}{1+e^{-x}}dx = -(1+e^{-x}) + \ln\left(1+e^{-x}\right),$$

so we have

$$\begin{aligned} y &= y_c + y_p = (c_1 - 1)e^x + (c_2 - 1)e^{2x} + (e^x + e^{2x}) \ln\left(1 + e^{-x}\right), \\ &= c_3 e^x + c_4 e^{2x} + (e^x + e^{2x}) \ln\left(1 + e^{-x}\right). \end{aligned}$$

Example (4)

Find the general solution of the differential equation

$$y''' + y' = \tan x \quad ; \quad 0 < x < \frac{\pi}{2}.$$

Solution

1) The the general solution of

$$y''' + y' = 0,$$

is

$$y_c = c_1 + c_2 \cos x + c_3 \sin x.$$

2) Let $y_1 = 1$, $y_2 = \cos x$ and $y_3 = \sin x$. The particular solution of the differential equation has the form

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3.$$

We have

$$W(x, y_1, y_2, y_3) = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = 1,$$

$$W_1(x, y_1, y_2, y_3) = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \tan x & -\cos x & -\sin x \end{vmatrix} = \tan x \ ,$$

$$W_2(x, y_1, y_2, y_3) = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \tan x & -\sin x \end{vmatrix} = -\sin x \ ,$$

$$W_3(x, y_1, y_2, y_3) = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \tan x \end{vmatrix} = \frac{-\sin^2(x)}{\cos x}.$$

Then we have,

$$u_1' = \frac{W_1}{W} = \tan x,$$

and

$$u_1 = \int \tan x dx = -\ln(\cos x) \ .$$

But

$$u_2' = \frac{W_2}{W} = -\sin x \ ,$$

then

$$u_2 = -\int \sin x dx = \cos x.$$

Also

$$u_3' = \frac{W_3}{W} = \frac{-\sin^2(x)}{\cos x},$$

hence,

$$u_3 = -\int \frac{\sin^2(x)}{\cos x}dx = -\int \frac{1-\cos^2(x)}{\cos x}dx = -\ln(\sec x + \tan x) + \sin x.$$

Thus,

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 + u_3y_3, \\ &= -\ln(\cos x) + \cos^2(x) - \sin x \ln(\sec x + \tan x) + \sin^2(x), \\ &= 1 - \ln(\cos x) - \sin x \ln(\sec x + \tan x). \end{aligned}$$

So the general solution of the differential equation is

$$\begin{aligned} y &= y_c + y_p = (c_1 + 1) + c_2 \cos x + c_3 \sin x - \ln(\cos x) \\ &- \sin x \ln(\sec x + \tan x) \end{aligned}$$

$$y = c_4 + c_2 \cos x + c_3 \sin x - \ln(\cos x) - \sin x \ln(\sec x + \tan x).$$

Example (5)

Find the solution of the initial value problem (IVP)

$$\begin{cases} 2x^2y'' + xy' - 3y = x^{-3} \quad ; \quad x > 0 \\ y(1) = 1 \quad , \quad y'(1) = -1. \end{cases}$$

Solution

1) We have to find the general solution of

$$2x^2y'' + xy' - 3y = 0.$$

By substituting $y = x^m$, we have

$$m(m-1) + m - 3 = (2m-3)(m+1) = 0,$$

hence the general solution of the homogeneous differential equation is

$$y_c = c_1\, x^{-1} + c_2\, x^{\frac{3}{2}}.$$

2) Let $y_1 = x^{-1}$, $y_2 = x^{\frac{3}{2}}$, then

$$y_p = u_1 y_1 + u_2 y_2.$$

We have

$$W(x, y_1, y_2) = \begin{vmatrix} x^{-1} & x^{\frac{3}{2}} \\ -x^{-2} & \frac{3}{2}x^{\frac{1}{2}} \end{vmatrix} = \frac{5}{2}x^{-\frac{1}{2}},$$

$$W_1(x, y_1, y_2) = \begin{vmatrix} 0 & x^{\frac{3}{2}} \\ \frac{1}{2}x^{-5} & \frac{3}{2}x^{\frac{1}{2}} \end{vmatrix} = -\frac{1}{2}x^{-\frac{7}{2}},$$

$$W_2(x, y_1, y_2) = \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & \frac{1}{2}x^{-5} \end{vmatrix} = \frac{1}{2}x^{-6}.$$

Then we have

$$u_1' = \frac{W_1}{W} = -\frac{1}{5}x^{-3},$$

and

$$u_1 = \frac{1}{10}x^{-2}.$$

Also we have

$$u_2' = \frac{W_2}{W} = \frac{1}{5}x^{-\frac{11}{2}},$$

hence

$$u_2 = -\frac{2}{45}x^{-\frac{9}{2}}.$$

So

$$y_p = u_1y_1 + u_2y_2 = \frac{1}{10}x^{-3} - \frac{2}{45}x^{-3} = \frac{1}{18}x^{-3}.$$

Then the general solution of the differential equation is

$$y = y_c + y_p = c_1\ x^{-1} \quad + c_2\ x^{\frac{3}{2}} + \frac{1}{18}x^{-3}.$$

We can obtain y_p by substituting $y_p = Ax^{-3}$, which implies $A = \frac{1}{18}$.

3)

$$y'(x) = -c_1 x^{-2} + \frac{3}{2} c_2 x^{\frac{1}{2}} - \frac{1}{6} x^{-4}.$$

From the conditions $y(1) = 1$ and $y'(1) = -1$, we deduce

$$c_1 + c_2 = \frac{17}{18},$$

and

$$-c_1 + \frac{3}{2} c_2 = -\frac{5}{6},$$

which implies $c_1 = \frac{9}{10}$ and $c_2 = \frac{2}{45}$. Thus the solution of the IVP is

$$y = \frac{9}{10} x^{-1} + \frac{2}{45} x^{\frac{3}{2}} + \frac{1}{18} x^{-3}.$$

Exercises

Use the variation of parameters method to find the general solution or initial value problems of the following differential equations.

- $y'' + y = \sec x; \quad 0 < x < \dfrac{\pi}{2}$
- $y'' - 2y' + y = \dfrac{e^x}{x}; \quad x > 0$
- $y'' - 12y' + 36y = e^{6x} \ln x; \quad x > 0$
- $y'' - 2y' + y = \dfrac{e^x}{(e^x + 1)^2}$
- $y'' - y = \dfrac{2}{\sqrt{1 - e^{-2x}}}$
- $y''' + 4y' = \sec 2x; \quad 0 < x < \dfrac{\pi}{4}$
- $2y''' - 6y'' = x^2$

- $y'' + y = \tan x; \;\; y\left(\frac{\pi}{3}\right) = 1, \;\; y'\left(\frac{\pi}{3}\right) = 0$
- $y'' + y = \sec^3(x); \;\; y(0) = 1, \;\; y'(0) = 1$
- $y'' - 2y' + y = \frac{e^x}{x}; \;\; y(1) = e, \;\; y'(1) = 0$

Power Series and Analytic Function

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Power Series and Analytic Function

- Some Reviews of Power Series
- Differentiation and Integration of a Power Series
- Power Series Solutions for Homogeneous Second-order Linear ODE with Nonconstant coefficients
- Ordinary Point and Singular Point
- Power Series Solutions about an Ordinary Point

A power series in $x - x_0$ is an infinite series of form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots, \qquad (1)$$

where the coefficients a_n are constants.

- The series (1) converges at the point $x = \alpha$ if

$$\lim_{n\to\infty} S_n(x) = \sum_{n=0}^{\infty} a_n (\alpha - x_0)^n$$

exists.

- The series (1) diverges at the point $x = \alpha$ if

$$\lim_{n\to\infty} S_n(x) = \sum_{n=0}^{\infty} a_n (\alpha - x_0)^n$$

does not exist.

Differentiation and Integration of a Power Series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

then

- $f'(x) = \sum_{n=0}^{\infty} a_n n (x - x_0)^{n-1}$, and

  $f''(x) = \sum_{n=0}^{\infty} a_n n(n-1)(x - x_0)^{n-2}$

- $\int f(x)\, dx = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+1}/(n+1)$

Power Series Solutions for Homogeneous Second-order Linear ODE with Nonconstant Coefficients

A general homogeneous second-order ODE has the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \tag{2}$$

which we will write in standard form

$$y'' + p(x)y' + q(x)y = 0, \tag{3}$$

where $p(x) = \frac{a_1(x)}{a_2(x)}$ and $q(x) = \frac{a_0(x)}{a_2(x)}$.

- A point $x = x_0$ is an **ordinary point** of the differential equation if $p(x)$ and $q(x)$ are analytic as $x = x_0$

- If $p(x)$ or $q(x)$ is not analytic at $x = x_0$ then we say that $x = x_0$ is a **singular point**.

Considering the defnitions of $p(x)$ and $q(x)$ above, we see that typically the points where $a_2(x) = 0$ are the singular points of the ODE.

Example

Locate the ordinary points, regular singular points and irregular singular points of the differential equation

$$(x^4 - x^2)y'' + (2x + 1)y' + x^2(x + 1)y = 0$$

Solution

We have $a_2(x) = x^4 - x^2$, $a_1(x) = 2x + 1$, $a_0(x) = x^2(x + 1)$, and so

$$a_1(x)/a_2(x) = \frac{2x + 1}{x^4 - x^2} = \frac{2x + 1}{x^2(x - 1)(x + 1)}$$

and

$$a_0(x)/a_2(x) = \frac{x^2(x + 1)}{x^4 - x^2} = \frac{1}{x - 1}.$$

We can see that every real number except $0, 1$ and -1 is an ordinary point the differential equation. To see which of the singular points $0, 1$ and -1 is a regular singular point and which is an irregular singular point for the differential.

we need to examine the two functions: $(x-x_0)a_1(x)/a_2(x)$, and $(x-x_0)^2a_0(x)/a_2(x)$ at the points $0, 1$ and -1.

At $x_0 = 0$, we have

$$(x-x_0)a_1(x)/a_2(x) = \frac{2x+1}{x(x-1)(x+1)},$$

and

$$(x-x_0)^2a_0(x)/a_2(x) = \frac{x^2}{x-1}.$$

The first function is not analytic at $x_0 = 0$, hence we conclude that $x_0 = 0$ is an irregular singular point.

At $x_0 = 1$, we have

$$(x-x_0)a_1(x)/a_2(x) = \frac{2x+1}{x^2(x+1)},$$

and

$$(x-x_0)^2a_0(x)/a_2(x) = x-1.$$

Since both of these expressions are analytic at $x_0 = 1$, we conclude that $x_0 = 1$ is a regular singular point.

Finally, for $x_0 = -1$, we have

$$(x - x_0)a_1(x)/a_2(x) = \frac{2x+1}{x^2(x-1)},$$

and

$$(x - x_0)^2 a_0(x)/a_2(x) = \frac{(x+1)^2}{x-1}.$$

Since both of these functions are analytic at $x_0 = -1$, we conclude that $x_0 = -1$ is a regular singular point for the differential equation.

Exercises

Locate the ordinary points, regular singular points and irregular singular points of the differential equation

- $xy'' - (2x-1)y' + y = 0$
- $x(x^2+1)^3 y'' + y' - 8xy = 0$
- $x^3(1-x^2)y'' + (2x-3)y' + xy = 0$

Power Series Solutions about an Ordinary Point

We now wish to find a series solution by expanding about an ordinary point $x = x_0$ of an ODE using the following method:

- Assume a solution of the form $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$
- Substitute the series into the ODE.
- Obtain an equation relating the coefficients, called a recurrence relationship.
- Apply any initial conditions.

Example (1)

Find the general solution of the differential equation

$$y' - 2xy = 0 \tag{4}$$

about the ordinary point $x_0 = 0$.

Solution It is clear that $x_0 = 0$ is an ordinary point since there are no finite singular points. The solution of (4) is of the form

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{5}$$

We have

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

then equation (4) becomes

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0 \tag{6}$$

We first make the same power of x as x^n in both series in (6) by letting $k = n - 1$ in the first series and $k = n + 1$ in the second one, we have

$$\sum_{k=0}^{\infty} (k+1) a_{k+1} x^k - \sum_{k=1}^{\infty} 2a_{k-1} x^k = 0. \tag{7}$$

We now let the index of summation starts by 1 in both series in (7), so that

$$a_1 + \sum_{k=1}^{\infty} \left[(k+1) a_{k+1} - 2a_{k-1} \right] x^k = 0. \tag{8}$$

For equation (8) to be satisfied, it is necessary that $a_1 = 0$ and

$$(k+1) a_{k+1} - 2a_{k-1} = 0, \quad \text{for all } k \geq 1. \tag{9}$$

Equation (9) provides a recurrence relation and we write

$$a_{k+1} = \frac{2a_{k-1}}{k+1} \quad \text{for all } k \geq 1 \tag{10}$$

Iteration of (10) then gives for $k = 1$

$$a_2 = a_0.$$

For $k = 2$

$$a_3 = \frac{2}{3}a_1 = 0.$$

For $k = 3$

$$a_4 = \frac{2}{4}a_2 = \frac{1}{2}a_0.$$

For $k = 4$

$$a_5 = \frac{2}{5}a_3 = 0.$$

And for $k = 5$

$$a_6 = \frac{2}{6}a_4 = \frac{1}{3!}a_0,$$

and so on.

Thus from the original assumption, we find

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots \\ &= a_0 \left(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots \right) \\ &= a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \quad \text{for all } x \in \mathbb{R}. \\ &= a_0 e^{x^2}. \end{aligned}$$

Example (2)

Solve the initial value problem by the method of power series about the initial point $x_0 = 0$.

$$\begin{cases} (1-x^2)y'' - xy' + 4y = 0 \\ y(0) = 1,\ y'(0) = 0 \end{cases} \tag{11}$$

Solution The two functions

$$a_1(x)/a_2(x) = \frac{-x}{1-x^2} = -\sum_{n=0}^{\infty} x^{2n+1} \quad \text{for } |x| < 1,$$

and

$$a_0(x)/a_2(x) = \frac{4}{1-x^2} = 4\sum_{n=0}^{\infty} x^{2n} \quad \text{for } |x| < 1,$$

are analytic for all $|x| < 1$,then the solution of the differential equation in (11) is given by

$$y = \sum_{n=0}^{\infty} a_n x^n \text{ for } |x| < 1.$$

Hence

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

for all $|x| < 1$. So we have

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + 4 \sum_{n=0}^{\infty} a_n x^n = 0. \quad (12)$$

Let $k = n - 2$ in the first series and $k = n$ in the other series, we get

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=2}^{\infty}k(k-1)a_kx^k - \sum_{k=1}^{\infty}ka_kx^k + 4\sum_{k=0}^{\infty}a_kx^k = 0.$$

All sums in (12) should start by the same index of summation 2, therefore we have

$$\sum_{k=2}^{\infty}\left[(k+2)(k+1)a_{k+2} - (k^2-4)a_k\right]x^k$$
$$+2a_2 + 4a_0 + (6a_3 + 3a_1)x = 0.$$

From this last identity, we conclude that

$$2a_2 + 4a_0 = 0, 6a_3 + 3a_1 = 0$$

and

$$a_{k+2} = \frac{(k^2-4)a_k}{(k+2)(k+1)}, \quad \text{for all } k \geq 2.$$

By using the initial conditions, we would observe that $a_0 = y(0) = 1$ and $a_1 = y'(0) = 0$, then $a_2 = -2$, $a_3 = 0$ and

$$a_{k+2} = \frac{k-2}{k+1} a_k, \quad \text{for all } k \geq 2.$$

So for $k = 2$,

$$a_4 = 0,$$

for $k = 3$,

$$a_5 = 0,$$

for $k = 4$,

$$a_6 = 0$$

for $k = 5$,

$$a_7 = 0,$$

for $k = 6$,

$$a_8 = 0 \text{ , and so on ,.....}$$

and so on. Then the initial value problem (11) has a unique solution given by

$$\begin{aligned} y = \sum_{n=0}^{\infty} a_n x^n &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots \\ &= 1 - 2x^2 + \ldots. \end{aligned}$$

for all $|x| < 1$.

Exercises

1. Find the general solution of the differential equation

$$y'' + y = 0$$

about the ordinary point $x_0 = 0$.

2. Solve the initial value problem by the method of power series about the initial point $x_0 = 0$.

$$\begin{cases} (1 - x^2)y'' - 2xy' + 6y = 0 \\ y(0) = 1,\ y'(0) = 0 \end{cases}$$

3. Compute the first four coefficients of power serises solution about the given initial point

$$\begin{cases} y'' - 2(x - 1)y' + 2y = 0 \\ y(1) = 1,\ y'(1) = 0 \end{cases}$$

Ordinary Differential Equation Systems with Constant Coefficients

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Introduction

- The D operator

Elimination Method

The D operator

Definition

A **differential operator** is an operator defined as a function of the differentiation operator. It is helpful, as a matter of notation first, to consider differentiation as an abstract operation, accepting a function and returning another (in the style of a higher-order function in computer science).

Here we will use the following notations:

$$D = d/dt, D^2 = d^2/d^2t, \ldots, D^n = d^n,$$

where t is the independent variable.

Simple Equivalents:

- D_u means $D_u \equiv \frac{du}{dt}$ but $uD \equiv u\frac{d}{dt}$.
- $D_y^2 \equiv D \times D_y \equiv \frac{d}{dt}\left(\frac{dy}{dt}\right)$.

- Similarly $D^2 \equiv \frac{d^2}{dt^2}$ and $D^3 \equiv \frac{d^3}{dt^3}$.

The following differential equation:

$$3\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 2y = 0,$$

may be expressed as:

$$(3D^2 + 7D + 2)y = 0$$

or

$$3D^2 + 7D + 2 = 0$$

Elimination Method

Definition

The elimination method consists in bringing the system of n differential equations into a single differential equation of order n.

In the elimination method you either add or subtract the system of differential equations into a single differential equation.

Example (1)

Find the general solution of the system

$$\begin{cases} \frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} + \frac{dx}{dt} - 3\frac{dy}{dt} - x + 2y = 0, \\ \frac{dx}{dt} + 2\frac{dy}{dt} + 2x - 4y = 0. \end{cases} \tag{1}$$

Solution We first write the system (1) in its operator form

$$\begin{cases} (D^2 + D - 1)[x] + (D^2 - 3D + 2)[y] = 0, \\ (D + 2)[x] + (2D - 4)[y] = 0. \end{cases} \tag{2}$$

To eliminate x, we apply the operator $D^2 + D - 1$ to the second equation in (2) and $D + 2$ to the first one and substrate the first from the second, we get

$$\left((D^2 + D - 1)(2D - 4) - (D + 2)(D^2 - 3D + 2)\right)[y] = 0,$$

or

$$\left(D^3 - D^2 - 2D\right)[y] = 0 \Leftrightarrow y''' - y'' - 2y' = 0. \tag{3}$$

The characteristic equation for (3) is

$$m^3 - m^2 - 2m = 0$$

whose roots are $0, 2, -1$. Thus

$$y(t) = c_1 + c_2 e^{2t} + c_3 e^{-t}.$$

Substitution of this last expression in the second equation of (1) gives

$$x' + 2x = 4c_1 + 6c_3 e^{-t}. \tag{4}$$

Equation (4) is a linear equation, we solve it to obtain

$$x(t) = 2c_1 + 6c_3 e^{-t} + c_4 e^{-2t}.$$

To eliminate the constant c_4 from the solution $x(t)$, we replace $x(t)$ and $y(t)$ in the first equation in (2) and we find $c_4 = 0$. Consequently

$$x(t) = 2c_1 + 6c_3e^{-t}.$$

Example (2)

Solve the system

$$\begin{cases} x'' + y' - 3x' + 2x - y = 0, \\ x' + y' - 2x + y = 0. \end{cases} \tag{5}$$

Solution

We first write the system (5) in its operator form

$$\begin{cases} (D^2 - 3D + 2)[x] + (D - 1)[y] = 0, \\ (D - 2)[x] + (D + 1)[y] = 0. \end{cases} \tag{6}$$

To eliminate y,we apply $(D + 1)$ to the first equation in (6) and $(D - 1)$ to the second and substrate the first from the second , we get

$$\left(D^3 - 3D^2 + 2D\right)[x] = 0 \Leftrightarrow x''' - 3x'' + 2x' = 0. \tag{7}$$

The general solution of (7) is

$$x(t) = c_1 + c_2 e^{2t} + c_3 e^{t}.$$

From the second equation in (5), we have

$$y' + y = c_3 e^t + 2c_1. \tag{8}$$

We solve the linear equation (8) to obtain

$$y(t) = 2c_1 + \frac{c_3}{2} e^t + c_4 e^{-t}.$$

To eliminate the extraneous constant c_4, we substitute for $x(t)$ and $y(t)$ in the first equation in (5) and find that $c_4 = 0$. Hence

$$y(t) = 2c_1 + \frac{c_3}{2} e^t.$$

Example (3)

Find the general solution of the system

$$\begin{cases} \frac{1}{2}x''' - y'' = \cos t, \\ \frac{1}{2}x'' + x + y' = -\cos t \end{cases} \tag{9}$$

Solution We write the system (9) in the operator form

$$\begin{cases} \frac{1}{2}D^3[x] - D^2[y] = \cos t, \\ (\frac{1}{2}D^2 + 1)[x] + D[y] = -\cos t. \end{cases}$$

To eliminate y, we apply the operator D to the second equation and then sum both equations

$$(D^3 + D)[x] = \sin t + \cos t \Leftrightarrow x''' + x' = \sin t + \cos t. \tag{10}$$

By using the method of undetermined coefficients method, we find that the general solution of equation (10) is given by

$$x(t) = c_1 + c_2 \cos t + c_3 \sin t - \frac{t}{2}(\cos t + \sin t). \tag{11}$$

Substitution of the expression (11) in the second equation of (9) gives

$$y(t) = \left(\frac{3}{2} + \frac{c_3}{2} - \frac{t}{4}\right)\cos t + \left(-\frac{1}{4} - \frac{c_2}{2} + \frac{t}{4}\right)\sin t - c_1 t + c_4.$$

To eliminate the extraneous constant c_4, we substitute for $x(t)$ and $y(t)$ in the second equation in (9) and find that $c_4 = 0$. Hence

$$y(t) = \left(\frac{3}{2} + \frac{c_3}{2} - \frac{t}{4}\right)\cos t + \left(-\frac{1}{4} - \frac{c_2}{2} + \frac{t}{4}\right)\sin t - c_1 t.$$

Exercises

Find the general solution of the following systems of differential equations

1. $$\begin{cases} x' + y - x = -t^2, \\ y' - x - 3y = 2t \end{cases}$$

2. $$\begin{cases} x'' - 3x' + y' + 2x - y = 0, \\ y' + x' - 2x + y = 0 \end{cases}$$

3. $$\begin{cases} y'' + x' + x = 0, \\ y' - y + x = \sin t \end{cases}$$

Fourier Series

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Fourier Series

- Orthogonal Function
- Fourier Series
  - Even and Odd Functions
  - Properties of Symmetric Functions
- Fourier Cosine and Sine Series
- Complex Form of a Fourier Series

Orthogonal Functions

Firstly, we will introduce a tool called inner product to define orthogonal functions and sets of orthogonal functions.

Definition

The inner product of two functions f and g on the interval $[\alpha, \beta]$ is the scalar (real number)

$$(f, g) = \int_{\alpha}^{\beta} f(x)g(x)\ dx.$$

Definition

We say that The two functions f and g are orthogonal functions on the interval $[\alpha, \beta]$ if

$$(f, g) = \int_{\alpha}^{\beta} f(x)g(x)\ dx = 0.$$

Example (1)

The two functions $f(x) = \cos x$ and $g(x) = \sin x$ are orthogonal on the interval $[-\pi, \pi]$ since

$$(f, g) = \int_{-\pi}^{\pi} \cos x . \sin x dx = 0.$$

Example (2)

The two functions $f(x) = x$ and $g(x) = e^{|x|}$ are orthogonal on any symmetric interval $[-A, A]$, where A is a positive real constant. By using integration by parts, It can be easily checked that

$$(f, g) = \int_{-A}^{A} x e^{|x|} dx = 0.$$

Definition

We say that The set of functions $\{\varphi_1(x), \varphi_2(x), \varphi_3(x), \ldots, \varphi_n(x), \ldots\}$ is orthogonal on the interval $[\alpha, \beta]$ if

$$(\varphi_n(x), \varphi_m(x)) = \int_\alpha^\beta \varphi_n(x)\varphi_m(x)\, dx = 0, \quad n \neq m.$$

Definition

We define the norm (length) of function f in terms of the inner product as the quantity

$$||f|| = \sqrt{(\varphi_n, \varphi_n)} = \left(\int_\alpha^\beta \varphi_n^2(x)\, dx\right)^{1/2}.$$

Definition

If $\{\varphi_1(x), \varphi_2(x), \varphi_3(x), \ldots, \varphi_n(x), \ldots\}$ is an orthogonal set of function on the interval $[\alpha, \beta]$ with the property $||\varphi_n|| = 1$ for $n = 1, 2, \ldots$, then the set $\{\varphi_n(x)\}_{n\geq 1}$ is said to be an orthonormal set on the interval.

$$(\varphi_n(x), \varphi_m(x)) = \int_\alpha^\beta \varphi_n(x)\varphi_m(x)\, dx = 0, \quad n \neq m.$$

Definition

A set of real-valued functions $\{\varphi_1(x), \varphi_2(x), \varphi_3(x), \ldots, \varphi_n(x), \ldots\}$ is said to be orthogonal with respct to weight function $w(x) > 0$ on the interval $[\alpha, \beta]$ if We define the norm (length) of function f in terms of the inner product as the quantity

$$(\varphi_n, \varphi_m)_{w(x)} = \int_\alpha^\beta w(x)\varphi_n(x)\varphi_m(x)\, dx = 0, \quad n \neq m.$$

Example (3)

Show that the set of functions $\{1, \sin x, \cos x, \sin 2x, \cos 2x, ..., \sin mx, \cos mx, ..\}$ is orthogonal on the interval $[-\pi, \pi]$. Find the corresponding orthonormal set on $[-\pi, \pi]$.

Solution We have to show that

$$\begin{aligned}
(1, \sin nx) &= 0,\ (1, \cos nx) = 0,\ (\sin nx, \sin mx) = 0, \\
(\cos nx, \cos mx) &= 0, (\sin nx, \cos mx) = 0, \quad \forall n \neq m.
\end{aligned}$$

$$(1, \sin nx) = \int_{-\pi}^{\pi} \sin nx dx = -\frac{1}{n} \cos nx|_{-\pi}^{\pi} = 0,$$

$$(1, \cos nx) = \int_{-\pi}^{\pi} \cos nx dx = \frac{1}{n} \sin nx|_{-\pi}^{\pi} = 0,$$

$$\begin{aligned}
(\sin nx, \sin mx) &= \int_{-\pi}^{\pi} \sin nx \sin mx dx \\
&= \int_{-\pi}^{\pi} \frac{\cos(n-m)x - \cos(n+m)x}{2} dx = 0,\ n \neq m,
\end{aligned}$$

$$
\begin{aligned}
(\cos nx, \cos mx) &= \int_{-\pi}^{\pi} \cos nx \cos mx dx \\
&= \int_{-\pi}^{\pi} \frac{\cos(n-m)x + \cos(n+m)x}{2} dx = 0, \ n \neq m, \\
(\sin nx, \cos mx) &= \int_{-\pi}^{\pi} \sin nx \ \cos mx \ dx \\
&= \int_{-\pi}^{\pi} \frac{\sin(n-m)x + \sin(n+m)x}{2} dx = 0.
\end{aligned}
$$

To determine the orthonormal set on $[-\pi, \pi]$, we have to divide each element by its norm.

$$\|1\|^2 = \int_{-\pi}^{\pi} dx = 2\pi,$$

$$\|\sin mx\|^2 = \int_{-\pi}^{\pi} (\sin mx)^2 dx = \int_{-\pi}^{\pi} \frac{1-\cos 2mx}{2} dx = \pi,$$
$$\|\cos mx\|^2 = \int_{-\pi}^{\pi} (\cos mx)^2 dx = \int_{-\pi}^{\pi} \frac{1+\cos 2mx}{2} dx = \pi.$$

Hence the orthonormal set on $[-\pi, \pi]$:

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, ..., \frac{\sin mx}{\sqrt{\pi}}, \frac{\cos mx}{\sqrt{\pi}}, .. \right\}.$$

Example (4)

Show that the functions

$$f(x) = 1, g(x) = 2x, h(x) = 4x^2 - 2$$

are orthogonal with respect to the weight function $w(x) = e^{-x^2}$ on the interval $(-\infty, \infty)$.

$$(1, 2x)_{w(x)} = \int_{-\infty}^{\infty} 2xe^{-x^2}dx = -2\int_{-\infty}^{\infty} e^{-x^2}dx = -2\, e^{-x^2}\Big|_{-\infty}^{\infty} = 0,$$

$$\begin{aligned}
(1, 4x^2 - 2)_{w(x)} &= \int_{-\infty}^{\infty} (4x^2 - 2)e^{-x^2}dx \\
&= -\int_{-\infty}^{\infty} 2xe^{-x^2}dx - 2\int_{-\infty}^{\infty} e^{-x^2}dx \\
&= -2xe^{-x^2}\Big|_{-\infty}^{\infty} + 2\int_{-\infty}^{\infty} e^{-x^2}dx - 2\int_{-\infty}^{\infty} e^{-x^2}dx \\
&= 0.
\end{aligned}$$

In the same way and by integration by parts, we find that

$$(2x, 4x^2 - 2)_{w(x)} = 0.$$

Exercises

1. Show that the set of the following functions are orthogonal on the given intervals.
   - $f(x) = \sin^2 x, \;\; g(x) = \cos x, \;\; [0, \pi]$
   - $f(x) = e^x, \;\; g(x) = xe^{-x} - e^{-x}, \;\; [0, 2]$
   - $f(x) = \cos x, \;\; g(x) = e^{-x}, \;\; \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$
2. Show that the set of the following functions are orthogonal with respect to the given weight function on the indicated interval.

$$f(x) = 1 - x, \;\; g(x) = \frac{1}{2}x^2 - 2x + 1; \;\; w(x) = e^{-x}, \;\; [0, \infty)$$

3. Determine the constants λ and δ so that the functions

$$f(x) = \lambda x + 2, \; g(x) = \delta x^2 - 6x + 1 \text{ and } h(x) = x - 1$$

are mutually orthogonal on $[0, 1]$ and then obtain the corresponding orthonormal set.

4. Show that the set of functions

$$\left\{1,\ \cos\frac{n\pi x}{L},\ \sin\frac{n\pi x}{L}\right\};\ m,n=1,2,3,\cdots;\ [0,L]$$

are orthogonal on the given interval and find the norm of each function.

Fourier Series

Theorem

Suppose that f and f' are piecewise continuous on the interval $[-T, T]$. Further, suppose that f is define outside the interval $[-T, T]$ so that it is periodic with period $2R$. Then f has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{T} + b_n \sin \frac{n\pi x}{T} \right).$$

Whose coefficients are given by

$$a_n = \frac{1}{T} \int_{-T}^{T} f(x) \cos \frac{n\pi x}{T} \, dx, \quad (n = 1, 2, \dots),$$

$$b_n = \frac{1}{T} \int_{-T}^{T} f(x) \sin \frac{n\pi x}{T} \, dx, \quad (n = 1, 2, \dots), \quad a_0 = \frac{1}{T} \int_{-T}^{T} f(x) \, dx.$$

Even and Odd Functions

Recall that if $f(x)$ is an even function then $f(-x) = +f(x)$.

Examples

$f(x) = x^4 - x^2$, $f(x) = (2-x)^2$ for $0 < x < 2$ and $f(x) = 2$ for $x \in [-a, a]$.

Recall that if $f(x)$ is an odd function then $f(-x) = -f(x)$

Examples

$f(x) = x^3$, $f(x) = x$.

Two symmetry properties of functions will be useful in the study of Fourier series. A function $f(x)$ that satisfies $f(-x) = f(x)$ for all x in the domain of f has a graph that is symmetric with respect to the $y - axis$. This function is said to be even. For example:

$$f(x) = \sqrt{2 + x^4}, g(x) = e^{-|x|},$$
$$h(x) = \cos x + \ln\left(1 + x^2\right),$$
$$k(x) = \begin{cases} |\sin x|, & |x| \leq \pi \\ 0, & |x| > \pi \end{cases}.$$

A function f that satisfies $f(-x) = -f(x)$ for all x in the domain of f has a graph that is symmetric with respect to the origin. It is said to be an odd function. For example:

$$f(x) = e^{|x|} \sin x,$$

$$h(x) = \sqrt{1+x^2} \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

$$k(x) = \begin{cases} x-1, & 0 < x < 1, \\ x+1, & -1 < x < 0, \\ 0, & |x| > 1 \end{cases},$$

$$M(x) = x^{1/3} - \sin x.$$

Properties of Symmetric Functions

- If $f(x)$ is an even piecewise continuous function on $[-L, L]$, then

$$\int_{-L}^{L} f(x)\ dx = 2\int_{0}^{L} f(x)\ dx$$

- If $f(x)$ is an odd piecewise continuous function on $[-L, L]$, then

$$\int_{-L}^{L} f(x)\ dx = 0$$

- For an even function, we have the Fourier coefficients

$$a_n = \frac{2}{T}\int_{0}^{T} f(x)\cos\frac{n\pi x}{T}\ dx, \quad (n = 1, 2, \dots),$$

$$a_0 = \frac{2}{T}\int_{0}^{T} f(x)\ dx,$$

and

$$b_n = 0, \quad (n = 1, 2, \dots)$$

- For an odd function, we have the Fourier coefficients

$$b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{n\pi x}{T}\,dx, \quad (n = 1, 2, \dots),$$

$$a_n = 0, \quad (n = 0, 1, 2, \dots)$$

- When n is an integer

$$\sin n\pi = 0 \quad \text{and} \quad \cos n\pi = (-1)^n.$$

Example (1)

Assume that there is a Fourier series converging to the function

$$\begin{aligned} f(x) &= \begin{cases} -x, & -T \le x < 0 \\ x, & 0 \le x \le T; \end{cases} \\ f(x+2T) &= f(x). \end{aligned}$$

Compute the Fourier series for the given function.

Solution The Fourier series has the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi x}{T} + b_n \sin \frac{n\pi x}{T}).$$

Since $f(-x) = f(x)\ \forall x \in [-T, T]$, then f is even on $[-T, T]$, hence $b_n = 0$, $(n = 1, 2, ...)$.
We compute to find that

$$a_0 = \frac{2}{T} \int_0^T f(x)dx = T,$$

$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T f(x)\cos\frac{n\pi x}{T}dx, \quad (n=1,2,...), \\ &= \frac{2}{T}\int_0^T x\cos\frac{n\pi x}{T}dx \\ &= \frac{2T}{(n\pi)^2}\left(\cos n\pi - 1\right), \quad (n=1,2,...), \end{aligned}$$

Thus the Fourier series for the function f is given by

$$f(x) = \frac{T}{2} - \frac{4T}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{T}.$$

Observe that from the obtained Fourier series, we can deduce that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

This follows from the fact that the Fourier series converges to $f(0)=0$ at $x=0$.

Example (2)

Find a Fourier series to represent the function

$$f(x) = x - x^2$$

from $x = -\pi$ to $x = \pi$. Deduce that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

We write

$$x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

We have

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) dx = \frac{-2}{3}\pi^2,$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx dx = -\frac{2}{\pi}\int_0^{\pi} x^2\cos nx dx \\ &= \frac{4}{n^2}(-1)^{n+1}, \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx dx \\ &= \frac{2}{n}(-1)^{n+1}. \end{aligned}$$

Hence

$$x - x^2 = \frac{-1}{3}\pi^2 - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx.$$

By setting $x = 0$, we obtain

$$\frac{-1}{3}\pi^2 - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = 0.$$

From which it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

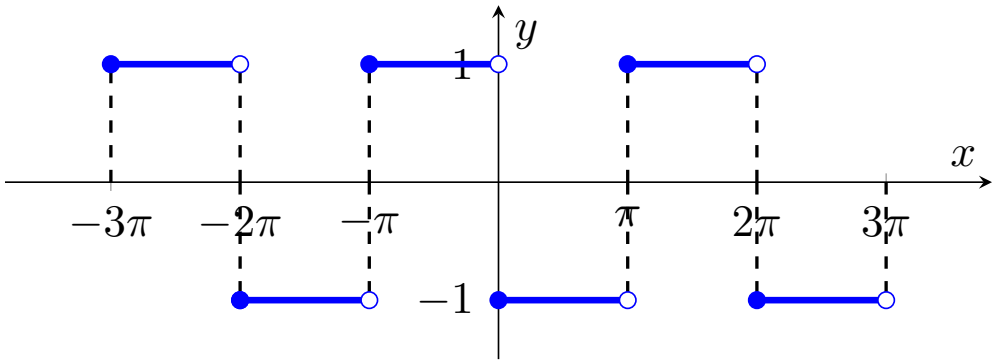
Example (3)

Let f be 2π-periodic function defined by

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ -1, & 0 \leq x < \pi. \end{cases}$$

Sketch the graph of f on $[-3\pi, 3\pi]$, find the Fourier series of f, and deduce the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$.

Solution We can see that f is an odd function on $(-\pi, \pi)$; $x \neq 0$



Thus,

$$\begin{aligned} a_n &= 0, \quad n = 0, 1, \cdots \\ b_n &= \frac{-2}{\pi} \int_0^{\pi} (1) \sin(nx) dx = \frac{x^2}{\pi} \left[\frac{\cos(nx)}{n} \right]_0^{\pi} = \frac{2}{n\pi} \left((-1)^n - 1 \right), \end{aligned}$$

hence, $n = 1, 2, \cdots$

$$\frac{f(x^+) + f(x^-)}{2} = \sum_{n=1}^{\infty} \frac{2}{n\pi} \left((-1)^n - 1 \right) \sin(nx), \ -\pi < x < \pi$$

At $x = \frac{\pi}{2}$, we have

$$f\left(\frac{\pi}{2}\right) = -1 = \sum_{n=1}^{\infty} \frac{2}{n\pi} \left((-1)^n - 1 \right) \sin\left(\frac{n\pi}{2}\right)$$

Note that $\sin\left(\frac{n\pi}{2}\right) = 0$, n is even.

Thus,

$$
\begin{aligned}
-1 &= \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi}\left((-1)^{2n-1}-1\right)\sin\left(\frac{(2n-1)\pi}{2}\right) \\
-1 &= \sum_{n=1}^{\infty} \frac{-4}{(2n-1)\pi}\sin\left(\frac{(2n-1)\pi}{2}\right) \\
-1 &= \frac{-4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi}{2}\right) \\
\frac{\pi}{4} &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}
\end{aligned}
$$

Exercises

1. Compute the Fourier series for the function

$$\begin{aligned} f(x) &= \begin{cases} \frac{1}{2}\pi + x, & -\pi \leq x < 0 \\ \frac{1}{2}\pi - x, & 0 \leq x \leq \pi; \end{cases} \\ f(x+2T) &= f(x) \text{ for all } x \in \mathbb{R}. \end{aligned}$$

Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

2. Obtain the Fourier series for the function

$$f(x) = \frac{1}{4}(\pi - x)^2,\ 0 < x < 2\pi,$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Fourier Cosine and Sine Series

Sometimes it is possible to represent a function as a Fourier Cosine or Sine Series.

To do this we use the properties of even and odd functions as defined previously.

To determine a series we usually extend the interval of definition to create a new function that is either even or odd depending on the type of series required.

If we require a Fourier cosine series then the new function created is chosen to be an even function. Similarly, If we require a Fourier sine series then the new function created is chosen to be an odd function.

For example, let $f(x)$ be defined on the interval $[0, L]$.

- If we require a Fourier cosine series then we create a new function created, $f_e(x)$, which is an even function over the interval $[-L, L]$. That is, we let

$$\begin{aligned} f_e(x) &= \begin{cases} f(x), & 0 < x < L, \\ f(-x), & -L \leq x \leq 0; \end{cases} \\ \text{with} \quad f_e(x+2L) &= f_e(x). \end{aligned}$$

- If we require a Fourier sine series then we create a new function created, $f_o(x)$, which is an odd function over the interval $[-L, L]$. That is, we let

$$f_o(x) = \begin{cases} f(x), & 0 < x < L, \\ -f(-x), & -L < x < 0, \end{cases}$$

and extending $f_o(x)$ to all x using the $2L$ periodicity.

Definition

Let $f(x)$ be piecewise continuous function on the interval $[0, L]$.

- The Fourier cosine series of $f(x)$ on $[0, L]$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx, \quad (n = 0, 1, 2, \dots).$$

- The Fourier sine series of $f(x)$ on $[0, L]$ is

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx, \quad (n = 1, 2, \dots).$$

Example (1)

Compute the Fourier sine series for the function

$$f(x) = \cos\frac{\pi x}{3},\ 0 < x < 3.$$

Solution We extend $f(x)$ as an odd function on $[-3, 3]$

$$f_o(x) = \begin{cases} \cos\frac{\pi x}{3}, & 0 \le x < 3, \\ -\cos\frac{\pi x}{3} & -3 \le x < 0. \end{cases}$$

The Fourier sine series representation of

$$f(x) = \cos\frac{\pi x}{3}$$

is

$$f(x) = \cos\frac{\pi x}{3} = \sum_{n=1}^{\infty} b_n \sin\frac{nx\pi}{3}, \quad 0 < x < 3,$$

where

$$\begin{aligned} b_n &= \frac{2}{3}\int_0^3 \cos\frac{\pi x}{3}\sin\frac{n\pi x}{3}dx \\ &= \frac{1}{3}\int_0^3 \left(\sin\frac{(n+1)\pi x}{3} - \sin\frac{(n-1)\pi x}{3}\right)dx \\ &= \begin{cases} 0, & n \text{ odd} \\ \frac{4n}{\pi(n^2-1)}, & n \text{ even} \end{cases} \end{aligned}$$

According to Fourier theorem, equality holds for $0 < x < 3$, but not at $x = 0$ and $x = 3$:

$$\cos\frac{\pi x}{3} = \frac{8}{\pi}\int_{n=1}^{\infty} \frac{n}{(4n^2-1)}\sin\frac{2nx\pi}{3}, \quad 0 < x < 3.$$

At $x = 0$ and $x = 3$, the Fourier series converges to

$$\frac{f(0^+) + f(0^-)}{2} = 0$$

and

$$\frac{f(3^+) + f(3^-)}{2} = 0,$$

respectively.

Example (2)

Compute the Fourier cosine series for the function

$$f(x) = e^{2x},\ 0 \le x \le 1.$$

and deduce that

$$\frac{2}{e^2 - 1} = \sum_{n=1}^{\infty} \frac{4}{4 + n^2\pi^2} \left[e^2(-1)^n - 1\right]$$

Solution We extend $f(x)$ as an even function on $[-1, 1]$

$$f_e(x) = \begin{cases} e^{2x}, & 0 < x < 1, \\ e^{-2x} & -1 < x < 0. \end{cases}$$

The Fourier cosine series representation of

$$f(x) = e^{2x},$$

is

$$f(x) = e^{2x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x, \quad 0 \le x \le 1,$$

where

$$a_0 = 2\int_0^1 e^{2x} dx = e^2 - 1,$$

$$\begin{aligned}
a_n &= 2\int_0^1 e^{2x} \cos n\pi x dx \\
&= 2\left[\frac{1}{2}\, e^{2x} \cos n\pi x\Big|_0^1 + \frac{1}{2} n\pi \int_0^1 e^{2x} \sin n\pi x dx\right] \\
&= e^2(-1)^n - 1 + n\pi\left[\frac{1}{2}\, n\pi e^{2x} \sin n\pi x\Big|_0^1 - \frac{1}{2} n\pi \int_0^1 e^{2x} \cos n\pi x dx\right] \\
&= e^2(-1)^n - 1 - \frac{1}{2} n^2\pi^2 \int_0^1 e^{2x} \cos n\pi x dx.
\end{aligned}$$

Hence

$$a_n = \frac{4}{4+n^2\pi^2}\left[e^2(-1)^n - 1\right]$$

The Fourier series is then

$$e^{2x} = \frac{e^2-1}{2} + \sum_{n=1}^{\infty} \frac{4}{4+n^2\pi^2}\left[e^2(-1)^n - 1\right]\cos n\pi x, \quad 0 \le x \le 1.$$

At $x = 0$, we have

$$\frac{1-e^2}{2} = \sum_{n=1}^{\infty} \frac{4}{4+n^2\pi^2}\left[e^2(-1)^n - 1\right].$$

Exercises

Test whether each of the following functions is odd or even, then expand it in a cosine or sine series.

- $f(x) = |\cos x|, \quad |x| < \pi.$
- $f(x) = x \cos x, \quad |x| < \pi.$
- $f(x) = x^2|x|, \quad |x| < 1.$
- $M(x) = \begin{cases} x+1, & -3 < x < 0 \\ -x+1, & 0 \le x < 3. \end{cases}$

Complex form of a Fourier Series

We have seen that Fourier Series in the interval $(-T, T)$ of a functon $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{T} + b_n \sin \frac{n\pi x}{T} \right).$$

Thus, from The Euler's formula we have the complex form of Fourier Series of f is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{T}},$$

where

$$c_n = \frac{1}{2T} \int_{-T}^{T} f(x) e^{\frac{in\pi x}{T}} \, dx.$$

Example

Obtain the complex form of the Fourier series for the function $f(x) = e^{\lambda x}$ $-\pi < x < \pi$ in the form

$$e^{\lambda x} = \frac{\sinh \lambda\pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{\lambda + in}{\lambda^2 + n^2} e^{inx},$$

and deduce that

$$\frac{\pi}{\lambda \sinh \lambda\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\lambda^2 + n^2}.$$

Solution We look for the coefficients c_n in the series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$,

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\lambda x}e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(\lambda-in)x}dx \\
&= \frac{1}{2\pi}\left[\frac{e^{(\lambda-in)\pi} - e^{-(\lambda-in)\pi}}{\lambda - in}\right] \\
&= \frac{1}{2\pi}\left[\frac{e^{\lambda\pi}(\cos n\pi - i\sin n\pi) - e^{-\lambda\pi}(\cos n\pi + i\sin n\pi)}{\lambda - in}\right] \\
&= \frac{1}{2\pi(\lambda - in)}\left(e^{\lambda\pi} - e^{-\lambda\pi}\right)\cos n\pi \\
&= \frac{1}{2\pi(\lambda - in)}(e^{\lambda\pi} - e^{-\lambda\pi})\cos n\pi \\
&= \frac{1}{2\pi(\lambda - in)}(2\sinh\lambda\pi)\cos n\pi \\
&= \frac{(-1)^n \sinh\lambda\pi}{\pi(\lambda - in)} = \frac{(-1)^n(\lambda + in)\sinh\lambda\pi}{\pi(\lambda^2 + n^2)}.
\end{aligned}
$$

Substituting this found c_n in the series to get

$$f(x) = e^{\lambda x} = \frac{\sinh \lambda\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n(\lambda + in)}{\lambda^2 + n^2} e^{inx}. \tag{1}$$

Now by setting $x = 0$ in (1), we obtain

$$\frac{\pi}{\sinh \lambda\pi} = \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{\lambda}{\lambda^2 + n^2} + i \frac{n}{\lambda^2 + n^2} \right).$$

By equating the real part, we have

$$\frac{\pi}{\lambda \sinh \lambda\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\lambda^2 + n^2}.$$

Exercises

Find the complex form of the Fourier series for:

- $f(x) = e^x, \quad |x| < \pi.$
- $f(x) = e^{-x}, \quad |x| < 1.$
- $f(x) = \cos(\theta x), \quad |x| < \pi.$

Fourier Integral

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Fourier Integral

- Fourier Series to Fourier Integral
- Fourier Cosine and Sine Series Integrals
- The Complex Form of Fourier Integral

Formula of Fourier Integral

The Fourier Integral of $f(x)$ defined on the interval $(-\infty, \infty)$ is given by

$$f(x) = \frac{1}{\pi}\int_0^\infty A(\lambda)\cos(\lambda x)\ d\lambda + \frac{1}{\pi}\int_0^\infty B(\lambda)\sin(\lambda x)\ d\lambda, \qquad (1)$$

where

$$A(\lambda) = \int_{-\infty}^{\infty} f(t)\cos(\lambda t)\ dt,$$

and

$$B(\lambda) = \int_{-\infty}^{\infty} f(t)\sin(\lambda t)\ dt.$$

Formula (1) can be written as

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\ dtd\lambda. \qquad (2)$$

Theorem

If f is absolutely integrable

$$\left(\int_{-\infty}^{\infty} |f(x)| \, dx < \infty \right),$$

and f, f' are piecewise continuous on every finite intreval, then Fourier integral of f converges to $f(x)$ at a point of continuity and converges to

$$\frac{f(x+0) + f(x-0)}{2}$$

at a point of discontinuity.

Example (1)

Express the function

$$f(x) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1, \end{cases}$$

as a Fourier integral. Hence evaluate $\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda$ and deduce the value of $\int_0^\infty \frac{\sin\lambda}{\lambda} d\lambda$.

Solution Since

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\lambda(t-x)dtd\lambda \\ &= \frac{1}{\pi}\int_0^\infty \int_{-1}^1 \cos\lambda(t-x)dtd\lambda \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_0^\infty \frac{\sin\lambda(t-x)}{\lambda}\Big|_{-1}^{1} d\lambda \\
&= \frac{1}{\pi}\int_0^\infty \frac{\sin\lambda(1-x)-\sin\lambda(-1-x)}{\lambda} d\lambda \\
&= \frac{1}{\pi}\int_0^\infty \frac{\sin\lambda(1+x)+\sin\lambda(1-x)}{\lambda} d\lambda \\
&= \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda} d\lambda.
\end{aligned}
$$

Hence

$$
\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda} d\lambda = \begin{cases} \frac{\pi}{2}, & |x|<1 \\ 0, & |x|>1, \end{cases}
$$

At $x=\pm 1$, $f(x)$ is discontinuous and the integral has the value $\frac{1}{2}(\frac{\pi}{2}+0)=\frac{\pi}{4}$.

Now by setting $x=0$, we have

$$
\int_0^\infty \frac{\sin\lambda}{\lambda} d\lambda = \frac{\pi}{2}.
$$

Example (2)

Compute the Fourier integral of the function

$$f(x) = \begin{cases} 0, & -\infty < x < -\pi \\ -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \\ 0, & \pi < x < \infty. \end{cases}$$

Solution We have

$$
\begin{aligned}
f(x) &= \frac{1}{\pi}\int_0^{\infty}\int_{-\pi}^{0} -\cos\lambda(t-x)dtd\lambda + \frac{1}{\pi}\int_0^{\infty}\int_0^{\pi}\cos\lambda(t-x)dtd\lambda \\
&= \frac{1}{\pi}\int_0^{\infty}\frac{\sin\lambda(t-x)}{\lambda}\Big|_{-\pi}^{0}d\lambda + \frac{1}{\pi}\int_0^{\infty}\frac{\sin\lambda(t-x)}{\lambda}\Big|_{0}^{\pi}d\lambda \\
&= -\frac{1}{\pi}\int_0^{\infty}\frac{-\sin\lambda x + \sin\lambda(\pi+x)}{\lambda}d\lambda \\
&+ \frac{1}{\pi}\int_0^{\infty}\frac{\sin\lambda(\pi-x)+\sin\lambda x}{\lambda}d\lambda \\
&= 2\int_0^{\infty}\frac{(1-\cos\lambda\pi)}{\lambda}\sin(\lambda x)d\lambda.
\end{aligned}
$$

This Fourier integral converges at the discontinuities points $-\pi, 0, \pi$ respectively to

$$\frac{f((-\pi)^+) + f((-\pi)^-)}{2} = \frac{-1}{2},$$

$$\frac{f(0^+) + f(0^-)}{2} = 0,$$

$$\frac{f(\pi^+) + f(\pi^-)}{2} = \frac{1}{2}.$$

Example (3)

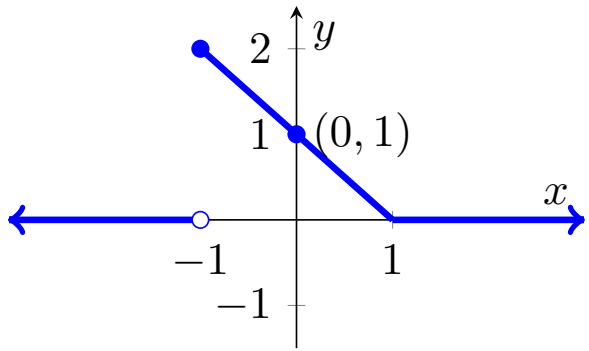
Consider the function

$$f(x) = \begin{cases} 0, & x < -1 \\ 1 - x, & -1 \leq x < 1 \\ 0, & x \geq 1. \end{cases}$$

Sketch the graph of f, find the Fourier integral and deduce the value of

$$\int_0^\infty \frac{\sin \lambda}{\lambda} d\lambda.$$

Solution



(1)

$$
\begin{aligned}
A(\lambda) &= \int_{-\infty}^{\infty} f(t)\cos(\lambda t)dt \\
&= \underbrace{\int_{-1}^{1} (1-t)\cos(\lambda t)dt}_{\text{by parts}} \\
&= \left[\frac{\sin(\lambda t)}{\lambda}(1-t)\right]_{-1}^{1} - \left[\frac{1}{\lambda^2}\cos(\lambda t)\right]_{-1}^{1} \\
&= \frac{-2}{\lambda}\sin(-\lambda) - \frac{\cos(\lambda)-\cos(-\lambda)}{\lambda^2} = \frac{2}{\lambda}\sin(\lambda)
\end{aligned}
$$

(2)

$$
\begin{aligned}
B(\lambda) &= \int_{-\infty}^{\infty} f(t)\sin(\lambda t)dt \\
&= \underbrace{\int_{-1}^{1}(1-t)\sin(\lambda t)dt}_{\text{by parts}} \\
&= \left[-\frac{\cos(\lambda t)}{\lambda}(1-t)\right]_{-1}^{1} - \left[\frac{1}{\lambda^2}\sin(\lambda t)\right]_{-1}^{1} \\
&= \frac{2}{\lambda}\cos(-\lambda) - \frac{\sin(\lambda)-\sin(-\lambda)}{\lambda^2} = \frac{2\cos(\lambda)}{\lambda} - \frac{2\sin(\lambda)}{\lambda^2}
\end{aligned}
$$

Thus,

$$\begin{aligned}
&\frac{f(x^+)+f(x^-)}{2}\\
&= \frac{1}{\pi}\int_0^\infty A(\lambda)\cos(\lambda x)d\lambda + \frac{1}{\pi}\int_0^\infty B(\lambda)\sin(\lambda x)d\lambda\\
&= \frac{1}{\pi}\int_0^\infty \left[\frac{2\sin\lambda\cos(\lambda x)}{\lambda} + \left(\frac{2\cos\lambda}{\lambda} - \frac{2\sin\lambda}{\lambda^2}\right)\sin(\lambda x)\right]d\lambda,
\end{aligned}$$

At $x = 0$, we have

$$\frac{f(0^+)+f(0^-)}{2} = \frac{1+1}{2} = 1 = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda}{\lambda}d\lambda,$$

hence,

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin\lambda}{\lambda}d\lambda$$

Exercises

Find the Fourier integral for the following functions

1. $$f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0. \end{cases}$$

2. $$f(x) = \begin{cases} 0, & -\infty < x < -2 \\ -2, & -2 < x < 0 \\ 2, & 0 < x < 2 \\ 0, & x > 2. \end{cases}$$

3. $$f(x) = \begin{cases} C, & |x| \leq 1 \\ 0, & |x| > 1, \end{cases}$$

   where C is a constant such that $C \neq 0$. Deduce the value of the integral $\displaystyle\int_0^\infty \frac{\sin\alpha}{\alpha} d\alpha$.

Fourier Cosine and Sine Series Integrals

The Fourier sine integral is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty C(\lambda) \sin(\lambda x) d\lambda,$$

where

$$C(\lambda) = \int_0^\infty f(t) \sin(\lambda t) dt.$$

The Fourier cosine integral is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty D(\lambda) \cos(\lambda x) d\lambda, \tag{3}$$

where

$$D(\lambda) = \int_0^\infty f(t) \cos(\lambda t) dt.$$

Example

Compute the Fourier integral of the function

$$f(x) = \begin{cases} |\sin x|, & |x| \leq \pi \\ 0, & |x| \geq \pi, \end{cases}$$

and deduce that

$$\int_0^\infty \frac{\cos \lambda\pi + 1}{1 - \lambda^2} \cos\left(\frac{\pi\lambda}{2}\right) d\lambda = \frac{\pi}{2}.$$

Solution We observe that the function f is even on the interval $(-\infty, \infty)$. So It has a Fourier cosine integral given by (3), that is

$$f(x) = \frac{2}{\pi} \int_0^\infty D(\lambda) \cos(\lambda x) d\lambda,$$

where

$$\begin{aligned} D(\lambda) &= \int_0^\infty f(t)\cos(\lambda t)dt = \int_0^\pi \sin t\cos(\lambda t)dt \\ &= \int_0^\pi \frac{\sin t(1-\lambda)+\sin t(1+\lambda)}{2}dt \\ &= \left.\frac{-\cos t(1-\lambda)}{2(1-\lambda)}\right|_0^\pi - \left.\frac{\cos t(1+\lambda)}{2(1+\lambda)}\right|_0^\pi \\ &= \frac{1}{1-\lambda^2}\left[\cos\pi\lambda+1\right]. \end{aligned}$$

Thus

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{1-\lambda^2}\left[\cos\pi\lambda+1\right]\cos(\lambda x)d\lambda. \tag{4}$$

Since f is continuous on the whole interval $(-\infty,\infty)$, the above integral converges to the given function $f(x)$. Setting $x=\pi/2$ in (4), we get

$$\int_0^\infty \frac{1}{1-\lambda^2}\left[\cos\pi\lambda+1\right]\cos\left(\frac{\lambda\pi}{2}\right)d\lambda = \frac{\pi}{2}.$$

Exercises

Find the Fourier sine and Fourier cosine integral for the following functions

1. $$f(x) = \begin{cases} x^2, & 0 < x \leq 10 \\ 0, & x > 10, \end{cases}$$

2. $$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ x+1, & 1 < x < 2 \\ 0, & x \geq 2. \end{cases}$$

The Complex Form of Fourier Integral

The complex form of Fourier integral is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \beta(\lambda) e^{-i\lambda x} \, d\lambda,$$

where

$$\beta(\lambda) = \int_{-\infty}^{\infty} f(t) e^{i\lambda t} \, dt$$

Example

Find the complex form of the Fourier integral for the function

$$f(x) = \begin{cases} e^x, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

Solution We have

$$\begin{aligned} \beta(\lambda) &= \int_{-\infty}^{\infty} f(t)e^{i\lambda t}dt = \int_{-1}^{1} e^{(i\lambda+1)t}dt \\ &= \frac{1}{(i\lambda+1)} e^{(i\lambda+1)t}\Big|_{-1}^{1} = \frac{1}{(i\lambda+1)}\left(e^{(i\lambda+1)} - e^{-(i\lambda+1)}\right) \\ &= \frac{1-i\lambda}{1+\lambda^2}\left[e^{(i\lambda+1)} - e^{-(i\lambda+1)}\right]. \end{aligned}$$

Hence

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1-i\lambda}{1+\lambda^2}\left[e^{(i\lambda+1)} - e^{-(i\lambda+1)}\right] e^{-i\lambda x}d\lambda.$$

Exercise

Find the complex form of the Fourier integral for the function

$$f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0. \end{cases}$$

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